MAT3004. Function spaces. MOCK-test 2. December 4th 2018.

Question 1

a) (1 point). Define what it means for a metric space $(X.d)$ to be compact.

b) (1 point). State what it means for a function $\omega : \mathbb{R}_+ \to \mathbb{R}$ to be a modulus of continuity for the function $f : X \to Y$ (X and Y are metric spaces).

c) (2 point). State the Arzela theorem for a set of functions $V \subset C[a, b]$ to be compact.

d) (2 point). State the Bessel inequality for orthonormal systems in an inner product space H.

Question 2 (4 points). Prove that any continuous function $f : X \to Y$ from a *compact* metric space X to a metric space Y is *uniformly* continuous.

Question 3 (2 points). Let $f \in H = L^2(-\pi, \pi)$. Prove that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

Hint: You may use without proving that $\{\frac{1}{\sqrt{\pi}} \sin nx\}_{n=1}^{\infty}$ is an orthogonal system in the space H.

Question 3 (2 points). Let $X := \mathbb{R}^2$ and let

$$F(x) := \begin{pmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

Is it possible to find a norm $\| \cdot \|_c$ on $\mathbb{R}^2$ in such a way that F will be a contraction on $(X, \| \cdot \|_c)$. Justify your answer.

Solutions

Question 1

a) The space (X, d) is compact if any sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

b) A function $\omega : \mathbb{R}_+ \to \mathbb{R}$ is a modulus of continuity of f if $\lim_{z\to 0} \omega(z) = 0$ and $d(f(x), f(y)) \le \omega(d(x, y))$ for all $x, y \in X$.

c) The set V is compact in $C[a, b]$ if 1) V is closed in $C[a, b]$; 2) V is bounded in $C[a, b]$; 3) equicontinuity: there exists $\omega : \mathbb{R}_+ \to \mathbb{R}$ which is the modulus of continuity for all $f \in V$.

d) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in H. Then, for any $x \in H$,

$$\sum_{n=1}^{\infty} (x, e_n)^2 \le \|x\|^2.$$

Question 2 Assume that f is not uniformly continuous. Then, there are two sequences x_n and z_n in X such that $d(x_n, z_n) < \frac{1}{n}$, but $d(f(x_n), f(z_n) \ge \varepsilon_0$ for some $\varepsilon_0 > 0$. Since X is compact, we may extract convergent subsequences from the sequences x_n and z_n: $x_{n_k} \to x_0$, $z_{n_k} \to z_0$. Then, on the one hand, since $d(x_n, y_n) < \frac{1}{n}$, we have $y_0 = x_0$. But on the other hand, since the function $(x, z) \to d(f(x), f(z))$ is continuous as a function from X^2 to $\mathbb{R}$, we have the convergence $d(f(x_{n_k}), f(z_{n_k})) \to d(f(x_0), f(z_0)) = 0$ and this contradicts the inequality $d(f(x_n), f(z_n)) \ge \varepsilon_0$. $\square$

Question 3. Let $e_n := \frac{1}{\sqrt{\pi}} \sin nx$. According to the hint, $\{e_n\}$ is an orthonormal system in H. Note also that $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \sqrt{\pi}(f, e_n)$. From the Bessel inequality we know that

$$\sum_{n=1}^{\infty} (f, e_n)^2 \le \|f\|_{L^2}^2 < \infty$$

and, therefore, $\lim_{n\to\infty}(f, e_n) = 0$.

Question 4. No! Assume that such norm exists, since all norms are equivalent in finite-dimensions, the space $(X, \|\cdot\|_c)$ is complete. The BCT guarantees that the contraction F has a unique fixed point in X. But F does not possess any fixed points. Indeed, the map F has a form $x_1 \to \frac{1}{2}x_1 + x_2 + 2$, $x_2 \to x_2 + 1$. So, for the second component p_2 of the fixed point p we have $p_2 = p_2 + 1$. $\square$