MAT3004. Function spaces. MOCK-test 2. December 10th 2019.

**Question 1**

**a) (1 point)**. Define what it means for a metric space $(X, d)$ to be totally bounded.

**b) (2 point)**. State the Hausdorff criterion for compactness of a metric space.

**c) (1 point)**. State the Cauchy-Schwarz inequality in a Hilbert space $H$.

**Question 2 (3 points)**. Let $f(x) := \frac{1}{x} - \frac{\sin(x)}{x^2}$. Does it belong to $L^2(-\pi, \pi)$? Justify your answer.

**Question 3 (5 points)**. Let $X := C[0, 1/2]$ and let the map $F : X \to X$ be defined as follows:

$$F(f)(x) := \frac{\pi}{2} + \int_0^x \sin(f(s))\, ds.$$

Prove that $F$ is a contraction (3 points) and find a fixed point of it (2 points).

*Hint:* $\frac{d}{dx} \ln(\tan(x/2)) = \frac{1}{\sin(x)}$.

## Solutions

**Question 1**

**a)** The space $(X, d)$ is totally bounded if for any $\varepsilon > 0$ there exists a covering of $X$ by finitely many $\varepsilon$-balls.

**b)** Hausdorff criterion: A metric space $(X, d)$ is compact if and only if it is complete and totally bounded.

**c)** Cauchy-Schwarz inequality: $|(x, y)| \le \|x\|\|y\|$ for any $x, y \in X$.

**Question 2** The function $f$ is *continuous* on $[-\pi, \pi]$ and by this reason $f \in L^p$ for every $p$. Indeed, the only point where singularity may appear is $x = 0$. Using that $\sin(x) = x - x^3/6 + o(x^5)$ near zero, we see that $f(x) = x/6 + o(x^3)$ and $\lim_{x\to 0} f(x) = 0$.

**Question 3**. We use that due to the mean value theorem, $|\sin(f) - \sin(g)| = |\cos(\xi)||f - g|$ and therefore, for any $f, g \in C[0, 1/2]$,

$$|F(f)(x) - F(g)(x)| \le |\int_0^x (\sin(f(s)) - \sin(g(s)))\, ds| \le$$

$$\le \int_0^{1/2} |f(s) - g(s)|\, ds \le \frac{1}{2}\|f - g\|_{sup}.$$

Taking the supremum over $x$, we see that $\|F(f) - F(g)\|_{sup} \le \frac{1}{2}\|f - d\|_{sup}$ and $F$ is a contraction.

The fixed point $y \in C[0, 1/2]$ of this map must satisfy

$$y(x) = \frac{\pi}{2} + \int_0^x \sin(y(s))\, ds.$$

From this equation we see that $y \in C^1[0, 1/2]$ and $y(0) = \frac{\pi}{2}$. Differentiating this equation in $x$, we get the following ODE:

$$y'(x) = \sin(y(x)),\ y(0) = \pi/2$$

Separating variables in this equation and using the hint, we get

$$\ln(\tan(y(x)/2) = x \quad \text{and} \quad y(x) = 2\arctan(e^x).$$