MAT3004. Function Spaces: Mock-Test 1. October 29th 2019

Question 1.

a) (2 points). Define what it means for a function $d : X^2 \to \mathbb{R}$ to be a *metric* on a set X.

b) (1 point). Define what norms on a vector space X are called *equivalent*.

c) (1 point). Give a definition of a Banach space.

Question 2. Give examples of metric spaces X and sets A and B such that

a) (2 points).

$$\partial(A \cap B) \not\subset \partial A \cap \partial B.$$

b) (2 points).

$$\partial A \cap \partial B \not\subset \partial(A \cap B).$$

Question 3. Let $X := l_\infty$ be a space of bounded sequences and let

$$\|a\|_1 := \sup_{n\in\mathbb{N}} \{|a(n)|\}, \quad \|a\|_2 := \sup_{n\in\mathbb{N}} \left\{\sin^2(n)|a(n)|\right\}.$$

a) (2 points). Prove that $\|a\|_2$ is a *norm* on X.

b) (3 points). Are the norms $\|a\|_1$ and $\|a\|_2$ on X equivalent? Justify your answer.

Hint: You may use without proving that

$$\liminf_{n\to\infty} \sin^2(n) = 0.$$

Solutions

Question 1

a) A function $d : X^2 \to \mathbb{R}$ is a metric on a set X if the following properties are satisfied:

1. Positivity: $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
2. Symmetry: $d(x, y) = d(y, x)$;
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$.

b) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there are positive constants C_1 and C_2 such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|, \quad \forall x \in X.$$

c) By definition, Banach space is a complete normed space.

Question 2.

a) For instance $X = \mathbb{R}$, $A = [0, 2]$, $B = [1, 3]$.

b) Take, e.g., $X = \mathbb{R}$, $A = (0, 1)$, $B = (1, 2)$.

Question 3.

a) Homogeneity and triangle inequality are obvious. Also $\|a\|_2 \geq 0$, so we only need to check that $\|a\|_2 > 0$ for $a \neq 0$. This follows from the fact that $\sin^2(n) \neq 0$ for all $n \in \mathbb{Z}$. Indeed, $\sin^2(x) = 0$ iff $x = \pi n$, $n \in \mathbb{Z}$ and since π is irrational there are no natural numbers in this list.

b) The norms are not equivalent. Indeed, using the hint, let us take a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $\sin^2(n_k) < \frac{1}{k}$. Take a sequence $a_k \in l^\infty$ by the following formula:

$$a_k(n) = \begin{cases} 0, & n \neq n_k \\ 1, & n = n_k. \end{cases}$$

Then, obviously $\|a_k\|_1 = 1$, but $\|a_k\|_2 = \sin^2(n_k) < \frac{1}{k} \to 0$ as $k \to \infty$ and the norms are not equivalent.