MAT3004. Function Spaces: Test 1. November 6th 2018

Question 1

a)(2 points). Define what it means for $d(x, y)$ to be a *metric* on a set X.

b) (1 point). Define what is an *interior* $\operatorname{int} V$ and a *closure* $\bar{V}$ of a set V in a metric space X.

c) (1 point). Define what metric spaces (X, d) are called *complete*.

Question 2 (4 points). Let $(\mathbb{R}, d)$ be a totally disconnected metric space (i.e., the set of real numbers endowed with the following metric: $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$). Is this space *separable*? Justify your answer.

Question 3 (4 points). Let $X := C[0, 1]$ and let

$$\|f\|_1 := \int_0^1 |f(x)|\, dx, \quad \|f\|_2 := \int_0^1 x|f(x)|\, dx.$$

Are these norms equivalent? Justify your answer.

Solutions

Question 1

a) A function $d : X^2 \to \mathbb{R}$ is a metric on a set X if the following properties are satisfied:

1. Positivity: $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
2. Symmetry: $d(x, y) = d(y, x)$;
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$.

b) A point $x \in \text{int}(V)$ if there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset V$. A point $x \in \bar{V}$ if there is a sequence $\{x_n\}_{n=1}^\infty \subset V$ such that $x_n \to x$.

c) The space (X, d) is complete if every Cauchy sequence in it is convergent.

Question 2. This space is not separable. Indeed, let $V := \{x_n\}_{n=1}^\infty$ be a countable dense set in it. Then, by the definition of the metric $B_{1/2}(x_n) \cap B_{1/2}(x_m) = \varnothing$ if $n \neq m$. The density of V then implies that $V = \mathbb{R}$ which is impossible since $\mathbb{R}$ is not countable.

Question 3. The norms are not equivalent. Indeed, the ratio of weights here is $Q(x) = x$ which vanishes at $x = 0$. This guesses what sequence of functions we may take for the counterexample. Namely, let

$$f_n(x) := \begin{cases} 1 - nx, & x \leq 1/n; \\ 0, & x > 1/n. \end{cases}$$

Then,

$$\|f_n\|_1 = \int_0^{1/n} (1 - nx)\, dx = 1/n \int_0^1 (1 - y)\, dy = \frac{1}{2n}$$

but

$$\|f_n\|_2 = \int_0^{1/n} x(1 - nx)\, dx = \frac{1}{n^2} \int_0^1 y(1 - y) = \frac{1}{6n^2}$$

and the norms are not equivalent.