Question 1

(a) Define what it means for a metric space (X, d) to be *totally bounded.* [1]

(b) State the Hausdorff criterion for a metric space (X, d) to be compact. [2]

(c) State the Banach contraction theorem. [2]

Question 2
Find the angle between functions $f(x) = x$ and $f(x) = x^3$ in a Hilbert space $H := L^2(-1, 1)$ with the standard inner product. [2]

Question 3
Let $f(x) = \sqrt{x^2 + 1}$. Is it a contraction on $\mathbb{R}$ with the standard norm? Justify your answer. [2]

Question 4
Let (X, d) be a compact metric space and let the map $f : X \to Y$ be continuous and bijective (Y is another metric space). Prove that the inverse map $f^{-1} : Y \to X$ is continuous. [4]

END OF TEST

INTERNAL EXAMINER: Prof S. Zelik

Solutions

Question 1.

1) A metric space (X, d) is totally bounded if for any $\varepsilon > 0$ there exists a finite covering $X = \cup_{k=1}^{N} B_\varepsilon(x_k)$ by finitely many ($N = N(\varepsilon)$) ε-balls in X.

2) Hausdorff criterion: A metric space X is compact if and only if X is complete and totally bounded.

3) Banach contraction theorem: Let X be a complete metric space and $F : X \to X$ be a contraction. Then there exists a unique fixed point $p \in X$ of the map F.

Question 2. The angle is computed via $\langle(f, g) = \cos^{-1} \frac{(f,g)}{\|f\|\|g\|}$. In our case $(f, g) = \int_{-1}^{1} x^4\, dx = \frac{2}{5}$, $\|f\|^2 = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$, $\|g\|^2 = \int_{-1}^{-1} x^6\, dx = \frac{2}{7}$ and $\langle(f, g) = \cos^{-1} \frac{\sqrt{21}}{5}$.

Question 3. The map is NOT a contraction. The simplest way to see that is to observe that there is no fixed points of this map and if it were a contraction the BCT would imply the existence of it. Can be proved also directly using the mean value theorem or even without it. Indeed, let $x, y \in \mathbb{R}$, then

$$|\sqrt{x^2+1} - \sqrt{y^2+1}| = |\frac{x^2 - y^2}{\sqrt{x^2+1} + \sqrt{y^2+1}}| = \frac{|x+y|}{\sqrt{x^2+1} + \sqrt{y^2+1}}|x - y|$$

and, for any $\kappa < 1$, the inequality $\frac{|x+y|}{\sqrt{x^2+1}+\sqrt{y^2+1}} \leq \kappa$ fails to hold for all $x, y \in \mathbb{R}$.

Question 4. Argue by contradiction. Let $f^{-1} : Y \to X$ be discontinuous at some $y_0 \in Y$. Then, there is a sequence $y_n \to y$ converging to y_0 such that $f^{-1}(y_n) := x_n \notin B_{\varepsilon_0}(x_0)$ for some $\varepsilon_0 > 0$. Let us utilize compactness of X which allows us to extract a convergent subsequence $x_{n_k} \to \bar{x}_0 \neq x_0$ in X. By continuity,

$$y_0 = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = f(\bar{x}_0).$$

But, by the construction $f(x_0) = y_0$ which contradicts the injectivity of f.