Lecture. Linear stability and perturbations. 2D case (continued)

Reminder: We study the linearized 2D Euler equation in the vorticity formulation near the radially symmetric vortex $\bar{w} = G(|x|)$, $G(s) \sim (1+s)^{-\alpha}$ when $s \to \infty$, $\alpha \in (0,2)$, G is smooth. The corresponding velocity profile is $\boxed{\bar{u}(x) = \frac{x^\perp}{|x|}\Omega(|x|)}$, $\boxed{\Omega(s) = \int_0^s r\, G(r)\, dr}$

Linearization:

$$\partial_t w = Lw,$$

$$\boxed{Lw = L_G w := -(\bar{u}, \nabla)w - (BS(w), \nabla)G = L_a w + L_c w}$$

$\boxed{BS(w) = \nabla^\perp(-\Delta)^{-1} w}$ - Biot-Savart operator

In polar coordinates: These operators become **diagonal** with respect to the angular variables: if

$\boxed{w = \sum_{k \in \mathbb{Z}}{}' w_k(R) e^{ik\varphi}}$ Then for $k \neq 0$

$$\boxed{\psi_k(s) = \Delta_k^{-1} w_k = \left(\left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{k^2}{R^2}\right)^{-1} w_k\right)(s) = \frac{-1}{2|k|} s^{|k|} \int_s^\infty w_k(r) r^{1-|k|}\, dr -}$$

$$\boxed{- \frac{1}{2|k|} s^{-|k|} \int_0^s w_k(r) r^{1+|k|}\, dr}$$

We conclude from the last formula that

$$W_k \in L^2(\mathbb{R}^2) \Rightarrow \Psi_k \in L^\infty(\mathbb{R}^2) \text{ if } k \neq \pm 1$$

But for $k = \pm 1$, Ψ_k are defined by modulo of constants only. To avoid these problems we drop out these modes and consider our problem in the space

$$\widetilde{L}_2(\mathbb{R}^2) := \{ w \in L^2(\mathbb{R}^2),\ W_k(R) \equiv 0,\ k = \pm 1 \}$$

Main result (from the previous lecture)

① The operator L_a is anti-self-adjoint in $\widetilde{L}_2$

② The operator L_c is compact in $\widetilde{L}_2$

$\Rightarrow$ $\sigma_{ess}(L) \subset i\mathbb{R}$ and we have only the isolated eigenvalues of the finite multiplicity outside of $i\mathbb{R}$

$\Rightarrow$ if we interested in exponentially unstable solutions, we need to consider the eigenvalue problem

$$(*) \qquad L_a w + L_c w = \lambda w,\ \operatorname{Re}\lambda > 0,\ w \in \widetilde{L}_2(\mathbb{R}^2)$$

We write out this equation in the Fourier base:

$$L_a(w)_k = -ik\,\Omega(1\times 1)\,w_k(1\times 1)$$ multiplication operator

$$L_c(w)_k = +ik\,\frac{G'(1\times 1)}{1\times 1}\,\psi_k(1\times 1) \qquad \Delta_k \psi_k = w_k$$

Denote: $\boxed{E(s) := \frac{G'(s)}{s}}$, $\boxed{\mu = i\frac{\lambda}{k}}$. Then (*) reads

$$-ik\,\Omega(s)\,w_k(s) + ik\,E(s)\,\psi_k(s) = \lambda\, w_k(s)$$

$$\Rightarrow \quad (**) \qquad \boxed{\Omega(s)\,w_k(s) - E(s)\,\psi_k(s) = \mu\, w_k(s)} \qquad \boxed{\operatorname{Im}\mu > 0}$$

NB! (Surprice!) (**) is actually independent on k

(althoug this dependence is hidden in Δ_k, $\Delta_k\psi_k = w_k$)

$$\boxed{w_k(s) = \frac{E(s)}{\Omega(s)-\mu}\,\psi_k(s)}$$ and returning back to the stream function

$$\boxed{\left(-\frac{d^2}{ds^2} - \frac{1}{s}\frac{d}{ds} + \frac{k^2}{s^2}\right)\psi(s) + \frac{E(s)}{\Omega(s)-\mu}\,\psi(s) = 0}$$

One more trick: change of variable $t = \ln s$

Reduces this equation to the version of

Rayleigh stability equation

(R) $$-\frac{d^2}{dt^2}\psi + k^2\psi + \frac{A(t)}{R(t)-M}\psi = 0 \qquad t \in \mathbb{R}$$

where $$A(t) = e^{2t}E(e^t), \quad R(t) = \Omega(e^t)$$

Returning back to the profile $G(|x|)$

$$R(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} G(e^\tau)\, d\tau, \qquad A(t) = R''(t) + 2R'(t)$$

$\Rightarrow$ all terms depend in a simple and explicit way on the initial profile $G(|x|)$

Classical Rayleigh equation (1880) - stability of shear flows:

(RC) $$-\frac{d^2}{dt^2}\psi + k^2\psi + \frac{u''(t)}{u(t)-\mu}\psi = 0$$

Here the 2D shear flow $\bar{U}(x,y) = (0, u(x))$ is considered in a finite or infinite strip (you need to change $x \to t$)

What is known about this equation?

After more than 100 years of intensive study (still a complicated problem!)

① The essential spectrum is generated by

$$\mu \in [\min R(t), \max R(t)]$$

(due to the singularity in the potential)

In our case $A(t) \to 0$ $t \to \pm\infty$ R is bounded

$R(t) \to 0$ $t \to +\infty$ $R(t) \to \frac{1}{2} G(0)$ $t \to -\infty$

⇒ essential spectrum of the non-linear spectral problem is exactly this interval.

② No unstable eigenvalues can bifurcate from the continuous spectrum, but there may be eigenvalues frozen in the continuous spectrum and they are responsible for instability. unstable eigenvalues may bifurcate from such frozen eigenvalues only.

Was "partially" known by Rayleigh (many years befor the spectral theory was developed)

Rayleigh criterion: for the instability at least one inflection point of $U(t)$ must exist.

In our case this reads: $\boxed{A(a)=0}$ for some $a \in \mathbb{R}$

Simple idea Behind:

Let (R) be satisfied for some $r>0$ and $\mu=\mu_R+i\mu_I$, $\mu_I\neq 0$ and some eigenfunction ψ. Let $\psi^*=\overline{\psi}$ be the adjoint eigenfunction. Then $\times$ (R) By $\overline{\psi}$, we get

$$\int_{\mathbb{R}} |\psi'|^2+k^2|\psi|^2\,dt+\int_{\mathbb{R}}\frac{A(t)(R(t)-\mu_r)}{(R-\mu_r)^2+\mu_i^2}|\psi|^2dt=-i\mu_I\int_{\mathbb{R}}\frac{A(t)}{(R-\mu_r)^2+\mu_I^2}|\psi|^2dt$$

$$\Rightarrow \boxed{\int_{\mathbb{R}}\frac{A(t)}{(R-\mu_r)^2+\mu_I^2}|\psi|^2dt=0} \Rightarrow \text{Rayleigh criterion}$$

Actually the last identity also gives us that instability cannot bifurcate from continuous spectrum.

Indeed assume that we have a sequence (k_n,μ_n), $\operatorname{Im}\mu_n>0$ accumulating to (k_0,μ_0), $k_0>1$, $\mu_0\in(\inf R(t),\sup R(t))$

Let $t_n\to t_0$, $R(t_n)=\operatorname{Re}\mu_n$ and let $A(t_0)\neq 0$

Let ψ_n be the corresponding eigenfunction, normalized by

$$\int_{\mathbb{R}}|\psi'|^2+k_n^2|\psi|^2dt\equiv 1$$

Then, using that $\|\psi_n\|_{L^2}\leq 1$ and the fact that $A(t_0)$ is sign-def in a neighbourhood of t_0

We get that

$$\int_{\mathbb{R}} \frac{|\psi_n(t)|^2}{|R(t)-M_n|^2}\,dt \leq C$$

$\Rightarrow$ $\frac{A(\cdot)}{R(\cdot)-M_n}\psi_n(\cdot) \in L^2(\mathbb{R})$ and uniformly bounded

+ since $A(t)\to 0$ $t\in\pm\infty$, we have uniform tail estimates

+ elliptic regularity for $\left(-\frac{d^2}{dt}+k_n\right)$ $\Rightarrow$

ψ_n are uniformly B-d in $H^2(\mathbb{R})$ + uniform tail estimates

$\Rightarrow$ compactnes $\Rightarrow$ $\psi_n \to \psi_0$ strongly in $H^1(\mathbb{R})$

Moreover $\frac{|\psi_n(t)|}{|R(t)-M_n|} \to \frac{|\psi_0(t)|}{|R(t)-M_0|}$ almost everywhere

$\Rightarrow$ weak convergence + $\int_{\mathbb{R}} \frac{|\psi_0(t)|^2}{|R(t)-M_0|^2}\,dt \leq C$ $\Rightarrow$ $\psi_0(t_0)=0$

Finally, $\psi_0(t)\not\equiv 0$ due to the normalization and satisfies the limit eigenvalue problem

$\Rightarrow$ frozen eigenvalue !

Remark: Although, only integer values of k are important for us, it is convenient to consider real values of k and dispersion curves $M=M(k)$. They are analytic

(Real analytic) until $K>0$ and $\mathrm{Im}\,\mu>0$

Analyticity follows from the fact that for any k, $\mathrm{Im}\,\mu\neq 0$ the multiplicity of the eigenvalue cannot be bigger than 1 (look at the Wronskian).

How the inflection points are connected with frozen eigenvalues?

If $A(t_0)=0$, we do not have singularity in the operator $\frac{A(t)}{R(t)-R(t_0)}\psi$ at least in the generic case of simple roots of $R(t)$, so we have the self-adjoint operator

$$-\frac{d^2}{dt^2}+\frac{A(t)}{R(t)-R(t_0)}$$

and we can find the negative eigenvalue $-m_0^2$ of this operator, say, by min-max if the negative part of the potential is big enough

$\Rightarrow$ frozen eigenvalue for problem (R)!

Moreover, it is possible to prove that any frozen eigenvalue can be constructed in such a way.

(will be discussed at the special lecture later)

Strategy :

(1) Most difficult: to do local bifurkation analysis and to find $G(s)$ such that the unstable branch indeed bifurcates from the properly chosen frozen eigenvalue

(2) Check that this branch can be extended till some integer value $k > 1$ (much easier, partially already done)

We have proved that it cannot return to the continuos spectrum if there are no proper frozen eigenvalues to return. We only need to check that

$$\frac{d}{dk} \operatorname{Re} M(k) > 0 \quad \text{if} \quad \operatorname{Im} M(k) > 0$$

(will be discussed at the special lecture later)

Now we just Assume that we are given an exponentially unstable eigenvalue of (R) for some integer k. After that we may cut-off $G(s)$ for large s

Lemma Let the assumption hold. Then we may cut-off the vorticity profile $G(s)$ making it zero for large s in such a way that the assumption remains true.

Proof: Perturbation arguments applied to (R).