Lecture: Applications to the initial hydrodinamical problem

Assume: We have found a vorticity profile $G(R)$ and the corresponding velocity profile $x^{\perp}\Omega(R)$ such that Ω is localized and the linearized operator

$$Lw = -x^{\perp}\Omega \cdot \nabla w - B\mathcal{S}(w)\cdot \nabla G$$

- Euler in the vorticity form.

$L_a w$ $\qquad$ $L_c w$

considered in a Hilbert space $\mathring{L}^2(\mathbb{R}^2)$ (without 1st and -1st angular harmonics) possesses an exponentially unstable eigenvalue λ, $\mathrm{Re}\lambda > 0$ (isolated, finite multi)

Recall: L, L_a, L_c are diagonal in the angular F-basis

$$(L_a w)_k = -ik\,\Omega(R)\,w_k(R), \qquad (L_c w)_k = ik\,\frac{G'(R)}{R}\,\psi_k(R)$$

$\psi_k = \Delta_k^{-1} w_k$ - stream function components

$\Rightarrow$ if w is an eigenvector with $\mathrm{Re}\lambda > 0$ then $w = w_k$ and

$$+\left(ik\,\Omega(R) + \lambda\right) w_k(R) = ik\,\frac{G'(R)}{R}\,\psi_k(R) \qquad k \neq \pm 1$$

Since Δ_k^{-1} increases smoothness and is invertible ($\mathrm{Re}\lambda > 0$)

$\Rightarrow$ $w = w_k$ is smooth and localized

We need to add the term $\Delta u + \frac{1}{2}(x\cdot\nabla + 1)u$ to the velocity equation. Perturbation analysis on the vorticity level is simpler:

$$\text{curl}\,(x_1\partial_{x_1}u + x_2\partial_{x_2}u) = \partial_{x_2}(x_1\partial_{x_1}u_1 + x_2\partial_{x_2}u_1) -$$

$$-\partial_{x_1}(x_1\partial_{x_1}u_2 + x_2\partial_{x_2}u_2) = x\cdot\nabla w + w$$

Thus the extra term in the vorticity form reads

$$\underbrace{\left(\Delta - \tfrac{1}{2}\right)w}_{\text{negatively def, self-adjoint}} + \underbrace{\tfrac{1}{2}(x\cdot\nabla + 2)w}_{\text{anti-self adjoint}} := L_{per}\,w$$

We need to solve the spectral problem

$$(L_{per} + L_a + L_c)w = \lambda w, \quad \text{Re}\,\lambda > 0 \qquad - (per\ 1)$$

solve = find the vorticity profile G such that such an eigenvalue exists

We know: Δ - is diagonal in the angular Γ-base, but $x\cdot\nabla$ is also diagonal. Indeed $\partial_{x_1}u = \partial_R u\,\frac{x_1}{R} - \partial_\varphi u\,\frac{x_2}{R^2}$

$$\partial_{x_2}u = \partial_R u\,\frac{x_2}{R} + \partial_\varphi u\cdot\frac{x_1}{R^2}$$

$x_1\partial_{x_1}u + x_2\partial_{x_2}u = R\partial_R u$ - we may study (per 1) in $\tilde{L}^2(\mathbb{R}^2)$ where L_c is compact.

Problem: No small parameters in (PER 1)! How to use the perturbation theory?

Trick: introduce the small parameter ε artifici

$$\varepsilon L_{per} w + L_a w + L_c w = \lambda(\varepsilon) w \qquad (\text{PER } \varepsilon)$$

- for $\varepsilon = 0$ we have a good isolated eigenvalue $\lambda(0) = \lambda$. if we prove that it persists for some small $\varepsilon_0 > 0$ (in a small neighbourhood of $\lambda(0)$), then we divide (PER ε) by ε and obtain a solution of (PER 1) for the new profile $G_\varepsilon(R) = \varepsilon^{-1} G(R)$ and $\gamma_\varepsilon = \varepsilon^{-1}\lambda(\varepsilon)$ $\Rightarrow$ it remains to study the perturbation problem (PER ε) = Check conditions of the Theorem

We take $A(\varepsilon) = \varepsilon L_{per} + L_a$ ① At $\varepsilon = 0$ we have an isolated eigenvalue $\lambda(0)$ by the assumption.

② $A(\varepsilon)$ is a "nice" operator (sum of self-adjoint negative definite and anti-self adjoint. Since $\mathrm{Re}\,\lambda > 0$, estimate $\| (A(\varepsilon) - \lambda_0)^{-1} \|_{\mathcal{L}^2(\mathbb{R}^2)} \leq C$ follows from the basic energy estimate

③ We take $\mathcal{D}$ = "smooth functions with polynomical decay rate"

The fact that $L_c \mathcal{D} \subset \mathcal{D}$ is obvious (follows from the explicit formulas and the continuity of matrix elements is also immediate

$\Rightarrow$ the persistence of the isolated eigenvalue is proved and the unstable vorticity profile for Navier-Stokes is constructed.

Remark: Arguing analogously we also get the instability in Euler equations (just drop out the operator Δu)

Next task: study bifurcations of unstable eigenvalues in Rayleigh equation from inflection points

(R) $$-\frac{d^2}{dt^2}\psi + k^2\psi + \frac{A(t)}{R(t)-\mu}\psi = 0, \quad R = \Omega(e^t), \quad A(t) = R'' + 2R$$

spectral problem for μ, instability corresponds to $\operatorname{Im}\mu > 0$

We proved unstable branches $\mu = \mu(k)$ can bifurcate from the "frozen" eigenvalues or the continuous spectrum $[\inf R, \sup R]$

We have also seen : inflection points $A(p)=0$ with $R'(p)\neq 0$ naturally generate such frozen eigenvalues: Indeed, let $\mu = R(p)$. Then, we have a self-adjoint operator

$$-\frac{d^2}{dx^2} + \frac{A(t)}{R(t)-R(p)}$$ - singularity disappears

Under very mild assumptions (basically the potential should have negative part), there exists a negative isolated simple eigenvalue $-k_0^2$ and positive eigenfunction ψ_0, such that

$$-\frac{d}{dt^2}\psi_0 + k_0^2\psi_0 + \frac{A(t)}{R(t)-R(p)}\psi_0 = 0$$

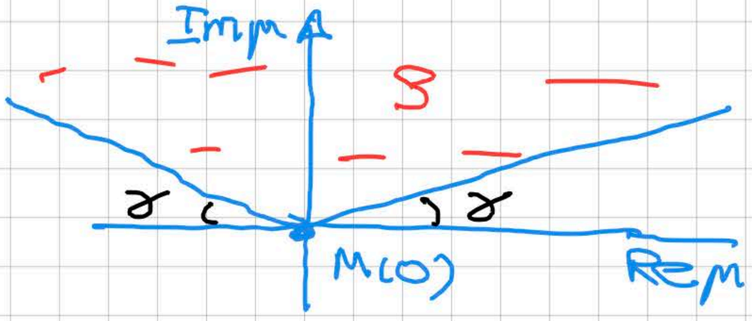
$\Rightarrow \{k_0, \mu = R(p)\}$ - frozen eigenvalue.

Q: What happens in a small neighbourhood of (k_0, μ)?

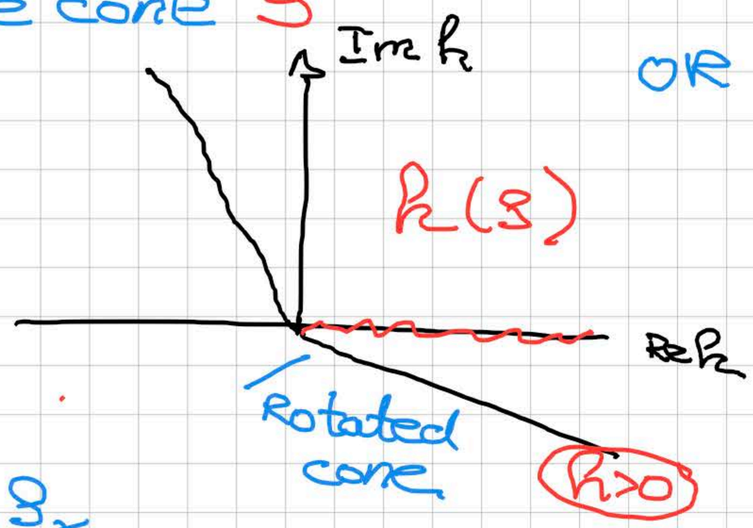
Normally: μ is a spectral parameter, $k_0^2 + \boxed{h}$ - perturb.

TRICK : Take μ as a perturbation parameter and consider $h \in \mathbb{C}$ as a spectral parameter and try to find $h = h(\mu)$ - allows to make the perturb REGULAR !!

Idea: Take $m \in \mathbb{C}$ from the cone S



$k = k(m)$ →



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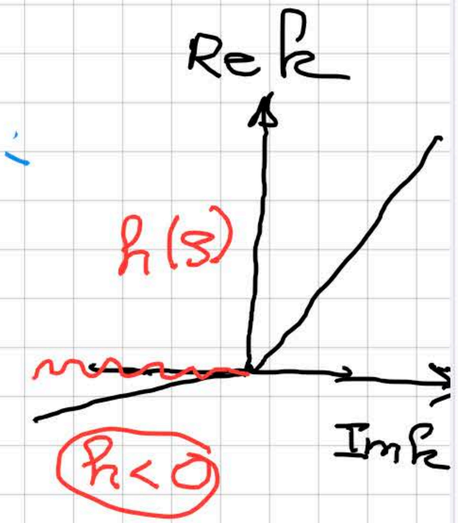
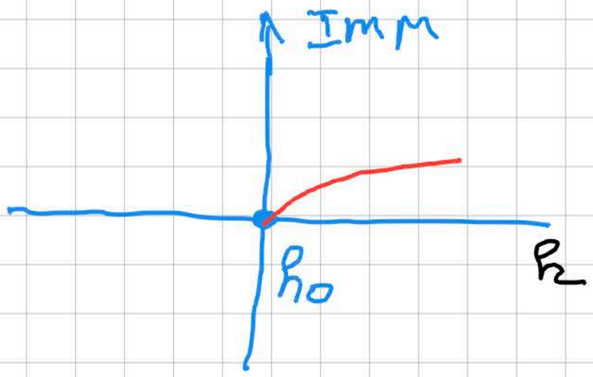


We need to prove that

1. k is regular enough in $S = S_\gamma$
2. Find the derivative at $m = m_0$ and prove that the rotation is not zero (a single quantity must be non-zero)

Surprice: This key quantity can be found explicitly

We will see that k is diffeo of these cone and the sign of this quattity tells us what case of two in the second picture is realized. This gives us the direction where the newly born branch $m = m(k)$ actualy 'Real goes



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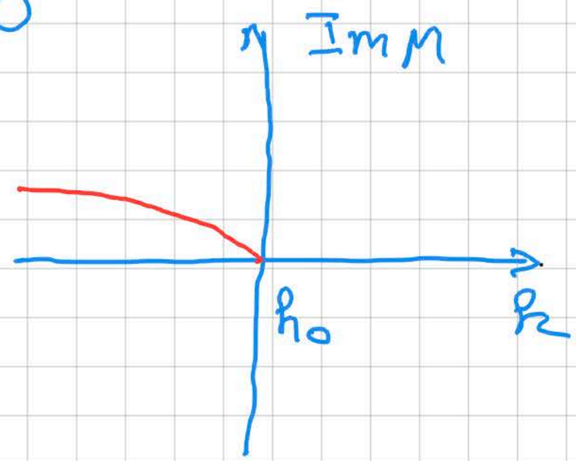

More details:

Spectral problem: $$-\frac{d^2}{dt}\psi_\mu + k_0^2\psi_\mu + \frac{A(t)}{R(t)-R(p)-\mu}\psi_\mu = k(\mu)\psi_\mu$$

Perturbation operator

$$B(\mu)\cdot\psi = \left(\frac{A(t)}{R(t)-R(p)-\mu} - \frac{A(t)}{R(t)-R(p)}\right)\psi.$$

Continuous at $\mu=0$ if μ is approaching ZERO inside the cone.

Distributional derivative:

$$\left\langle \frac{d}{d\mu}B(\mu)\psi, \Sigma\right\rangle\Big|_{\mu=0} = \lim_{\substack{\mu\to 0\\ \mu\in S}} \left\langle \frac{A(t)}{(R-R(p)-\mu)^2}\psi, \Sigma\right\rangle$$

How to compute this? Analogously to **Sokhotski-Plemelj formula:**

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon} = V.p.\left(\frac{1}{x}\right) \mp \pi i\delta(x)$$

Arguing analogously, we get

$$\left\langle \frac{d}{d\mu}B(\mu)\psi,\Sigma\right\rangle = V.p.\int_{\mathbb{R}}\frac{A(t)}{(R(t)-R(p))^2}\psi(t)\Sigma(t)\,dt + \pi i\,\frac{A'(p)}{(R'(p))^2}\psi(p)\Sigma(p)$$

NB! $B'(0)$ is not a bounded operator from $L^2\to L^2$ but acting from $H^1\to H^{-1}$.

and we may easily prove that $B(\mu)$ is C^1 in the cone as an operator from $H^{+1}(\mathbb{R}) \to H^{-1}(\mathbb{R})$

This is enough since we can write our spectral problem in the form

$$\psi + \left(-\frac{d}{dt^2} + k_0^2\right)^{-1/2} \frac{A(t)}{R(t)-R(p)-\mu} \left(-\frac{d}{dt^2} + k_0^2\right)^{-1/2} \psi = k \left(-\frac{d}{dt^2} + k_0^2\right)^{-1} \psi$$

and here all operators are bounded and C^1.

+ we know that the eigenvalue k_0^2 - is simple.

$\Rightarrow \exists\ k = k(\mu)$ - C^1-smooth in the cones.

The value of the derivative can be computed by the formula derived above

$$\frac{d}{d\mu} k(\mu)\Big|_{\mu=0} = \left(\frac{d}{d\mu} B(\mu)\psi_0, \psi_0\right)\Big|_{\mu=0} =$$

$$= \underbrace{V.P. \int_{\mathbb{R}} \frac{A(t)}{(R(t)-R(p))^2} \psi_0(t)^2 dt}_{\text{some unknown real constant}} + \pi i \underbrace{\frac{A'(p)}{R'(p)^2} \psi_0(p)^2}_{\text{crucial term which gives us instability!}}$$

The term $\pi i \frac{A'(p)}{R'(p)^2} \psi_0(p)^2$ is not zero if $A'(p) \neq 0$. Indeed, $\psi_0(t)$ is the eigenfunction which corresponds to the minimal eigenvalue of Schrödinger operator + min-max

$\Rightarrow$ $\psi_0(t)$ is positive $\Rightarrow$ $\psi_0(p) \neq 0$

Rotation of S_γ is always not-zero in our case!

Inflection point $\Rightarrow$ starting or ending point of an unstable branch

This is the key tool for constructing the unstable eigenvalue for linearized Euler

(will be considered in a separate lecture)