Lecture. Instability in Rayleigh equation (completion)

$$(R)\quad \boxed{-\frac{d^2}{dx^2}\psi + k^2\psi + \frac{A(t)}{R(t)-M}\psi = 0}$$

$A(t) = R'' + 2R$

R - vorticity profile

$R(-\infty) = C > 0$, $R(+\infty) = 0$

$R'(t)$ exp. decaying $t \to \pm\infty$

$M \in \mathbb{C}$ - spectral parameter

$\operatorname{Im} M > 0$ corresponds to instability

What we have already proved:

① Unstable eigenvalues can bifurcate from the "frozen eigenvalues" on the real line

② Any simple inflection point $A(x_0) = 0$, $A'(x_0) \neq 0$ ($R'(x_0) \neq 0$) generates a C^1-smooth branch of unstable eigenvalues of algebraic multiplicity 1.

$$\frac{d}{dM} k^2(M)\Big|_{\substack{M=0\\ I}} = \text{V.p.} \int_{\mathbb{R}}' \frac{A(x)}{(R(x)-R(x_0))^2} |\psi_0(x)|^2 dx +$$

$$+ \pi i \operatorname{sgn} R'(x_0) \cdot \frac{A'(x_0)}{R'(x_0)^2} |\psi_0(x_0)|^2 \neq 0$$

$\Rightarrow$ formular for $\frac{dM}{dk}(k) = \dots$

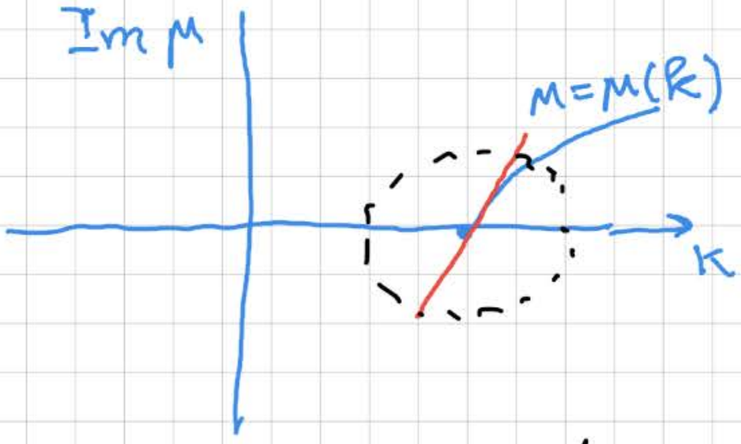
Reminder: ① follows from 1st nice formular for $M_I = \mathrm{Im}\,\mu > 0$ By multiplication of (R) by $\overline{\Psi}$

$$\int' |\Psi'|^2 + k^2 |\Psi|^2 dx = - \int' \frac{A(x)}{R(x) - \mu} |\Psi|^2 dx$$

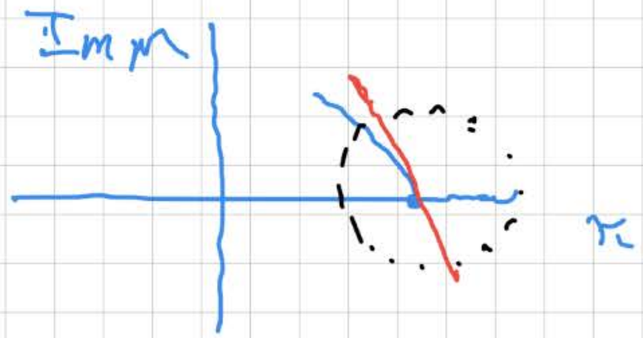
$$\Rightarrow \mathrm{Im} \int' \frac{A(x)}{R(x)-\mu} |\Psi^2(x)| dx = M_I \cdot \int' \frac{A(x)}{|R(x)-\mu|^2} |\Psi(x)|^2 dx = 0$$

In particular $\Rightarrow$ Rayleigh instability criterion $\Rightarrow$ how inflection points come into play!

from (2) we have two possible scenarios near the simple inflection point:



OR



depending on the sign of $R'(x_0) A'(x_0)$

Robust w.r.t. smooth changes of the profile $R(x)$ - straightforward calculations!

∃ 2nd general nice formula:

Let $\mathrm{Im}\, M > 0$ and let $\boxed{\Psi(x) = (R(x) - M)\,\eta(x)}$

Then, (R) reads:

$$(RR)\quad \boxed{-\frac{d}{dx}\left((R-M)^2 \frac{d}{dx}\eta\right) + k^2 (R-M)^2 \eta + 2R'(R-M)\eta = 0}$$

especially simple form for shear flows where $A = R''$

$$\boxed{-\frac{d}{dx}\left((R-M)^2 \frac{d}{dx}\eta\right) + k^2 (R-M)^2 \eta = 0}$$

multiplying (RR) by $\bar{\eta}$ and integrating by parts

$$\Rightarrow \boxed{\int (R-M)^2 \left(|\eta'|^2 + k^2 |\eta|^2 - \eta\bar{\eta}' - \bar{\eta}\eta'\right) dx = 0} \quad (\Pi)$$

Important! By Cauchy-Schwarz

$$\boxed{Q(\eta) = |\eta'|^2 + k^2 |\eta|^2 - \eta\bar{\eta}' - \bar{\eta}\eta' \geq 0 \quad \forall x}$$

if $\boxed{k \geq 1}$

Corollary (Howard's semi-circle theorem)

Let $\operatorname{Im} \mu > 0$ and $k \geq 1$. Then

$$(*) \qquad \left(\mu_R - \frac{R_{max} + R_{min}}{2}\right)^2 + \mu_I^2 \leq \left(\frac{R_{max} - R_{min}}{2}\right)^2$$

In particular

$$R_{min} < \mu_n < R_{max}$$

$\Rightarrow$ unstable branch may bifurcate from the singular spectrum $[R_{min}, R_{max}]$ only

Proof: Taking real and imaginary parts of (II) $\Rightarrow$

$$\int ((R-\mu_R)^2 - \mu_I^2)\, Q\, dx = 0 = \mu_I \int (R-\mu_R)\, Q\, dx = 0$$

$$\Rightarrow \int R\, Q\, dx = \mu_R \int Q\, dx \Rightarrow \int (R^2 - 2\mu_R^2 + \mu_R^2 - \mu_I^2)\, Q\, dx = 0$$

$$\Rightarrow \int R^2 Q\, dx = (\mu_R^2 + \mu_I^2) \int Q\, dx \Rightarrow$$

$$0 \geq \int (R - R_{max})(R - R_{min})\, Q\, dx = \int (\mu_R^2 + \mu_I^2 - (R_{max} + R_{min})\mu_R + R_{max} R_{min})\, Q\, dx \xrightarrow{Q \geq 0}$$

$$\mu_R^2 + \mu_I^2 - (R_{max} + R_{min})\mu_R + R_{max} R_{min} \leq 0$$

$\Rightarrow (*)$ $\square$

Nice formula (II) can be used to describe the structure of "frozen eigenvalues"

Corollary: Let $k \geq 1$ and R is monotone ($R'(x) \neq 0$)

Assume that $\mu = R(x_0)$ is the frozen eigenvalue which generates instability

Then $A(x_0) = 0 \Rightarrow$ inflection point.

Proof: Let ψ be the corresponding eigenfunct

$\Rightarrow$ (nice f-la 1) $\Rightarrow$ $\frac{\psi}{|R(x)-\mu|} \in L^2$

Assume $A(x_0) \neq 0$

$\Rightarrow$ $\psi(x)$ is smooth and $\psi(x_0) = 0$

idea: To justify (II) for the function

$\eta(x) = \psi(x)/(R(x)-R(x_0))$ - good function everywhere except of $x = x_0$

$\Rightarrow$ need to justify integration by parts at (x_0, ∞) and $(-\infty, x_0] \Leftrightarrow$

$$\lim_{x \to x_0} \left((R(x)-R(x_0))^2 \eta(x) \eta'(x) \right) = 0$$

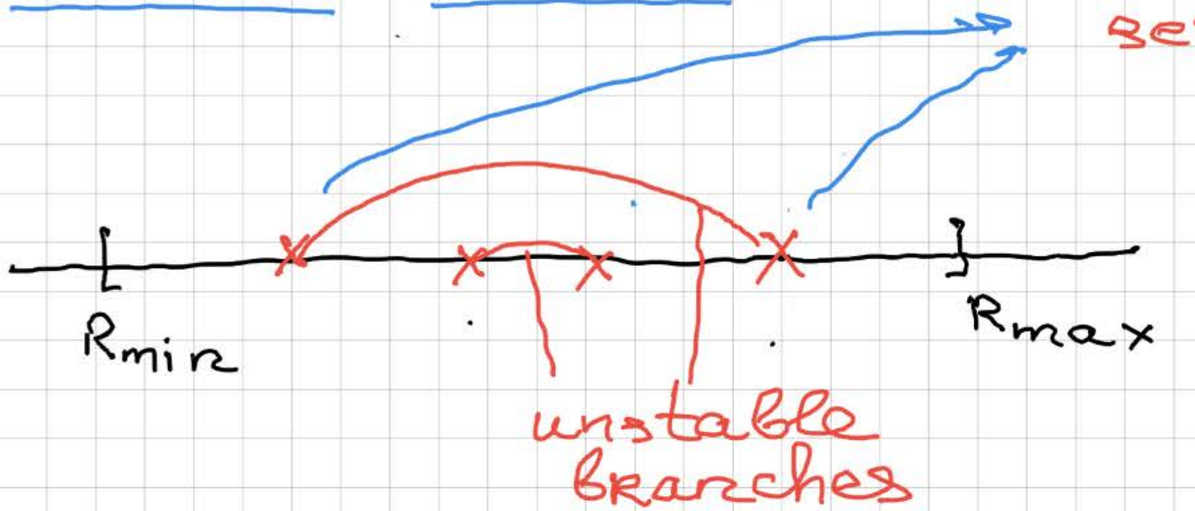
- straight forward calculation

$\Rightarrow$ nice f-la (II) holds $\Rightarrow$ $\eta(x) \equiv 0$

□

Remark: monotonicity assumption is not necessary. The result holds for general profiles $R(x)$.

General picture:



several inflection points for every of them there are finite number of k_3 where frozen eigenvalue exists

Usually enough for case study: you need to analyse finitely many bifurcations and look at the corresp. branches.

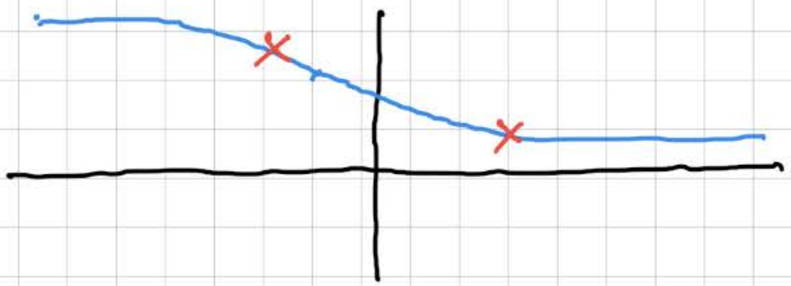
Problems: (1) We need to find instability for integer values of k (not all branches can be continued till integer k_3).

(2) Linear stability does not imply the non-linear one.

(Big problem, but we need not to consider it here)

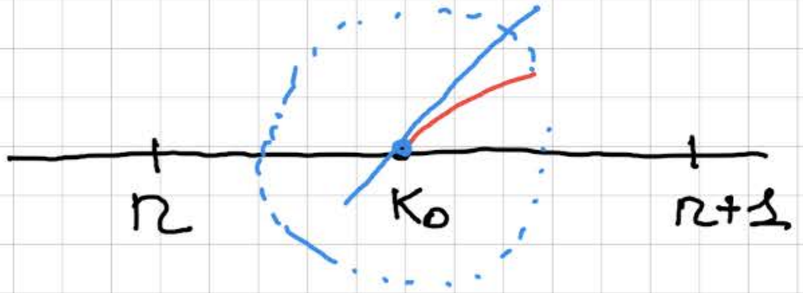
Finally, we return to the construction of the unstable profile $R(x)$.

(1) It is extremely easy to construct R with inflection points.



- this is for the case $A = R''$
⇒ two inflection points are not avoidable, for our case $A = R'' + 2R'$
- slightly more complicated

(2) From local bifurcation analysis:



- everything depends continuously on R, $K_0 \in \mathbb{R}_+$ is the least eigenvalue of

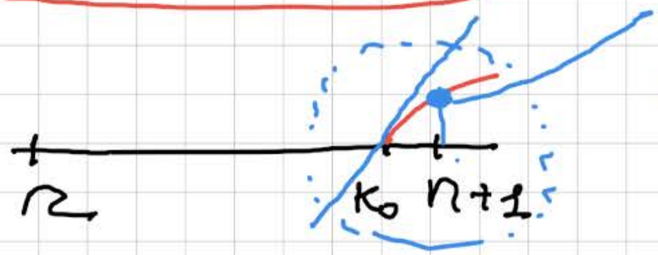
$$-\frac{d^2}{dx^2} + \frac{A(x)}{R(x) - x_0} := L_R$$

$1 < R$ - is integer K_0 may be not integer and the local neighbourhood may be small ⇒ a priory not extendable till $K = R+1$.

(3) Perturb R in such a way that $K_0(R_1) > R+1$ and take a continuous curve $R_t = (1-t)R_1 + tR$

controlled by min-max

can be made wit monotonicity (not necessary)



instability

everything is done!!