Lecture 2. Linear stability analysis for 2D-Euler equations.

2D Euler in $\mathbb{R}^2$.

$$(EN) \qquad \begin{cases} \partial_t u + (u,\nabla)u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases}$$

Equilibrium: $\bar{u}(x)$ - something we want to construct artificially

Linearization:

$$(EL) \qquad \partial_t v = L_{\bar{u}} v := -\Pi[(\bar{u},\nabla)v] - \Pi[(v,\nabla\bar{u}]$$

where Π is Leray projector

since $\Omega = \mathbb{R}^2$ $\quad \Pi : W^{s,p}(\mathbb{R}^2) \to W^{s,p}(\mathbb{R}^2)$, $1<p<\infty$.

0-order pseudo-diff operator

Vorticity formulation: $\quad w = \operatorname{curl} u := \partial_{x_2} u_1 - \partial_{x_1} u_2$

satisfies

$$(VN) \qquad \partial_t w + (u,\nabla)w = \operatorname{curl} f$$

where v can be found if w is known By the Biot-Savart law

$u = BS(w) = \nabla^{\perp}(-\Delta)^{-1} w$ so (VN) can be rewritten in the closed form

$$\partial_t w + (BS(w),\nabla)w = \operatorname{curl} f$$

Linearization: $G(x) = \operatorname{curl} \bar{u}(x)$

(VL) $\partial_t w = L_G w := -(\bar{u}, \nabla) w - (BS(w), \nabla) G = L_a w + L_c w$

we will see that (in our particular case) L_a - will be anti-Hermitian and L_c is compact (in the properly chosen spaces)

Stability analysis for Euler equations is an extremely classical topic studied from 19th century till nowadays. Still difficult and not properly understood. The case of spherically symmetric vortices is also classical and also not entirely understood. will not attempt to survey this topic here. Just mention

S. Friedlander, W. Strauss, M. Vishik, Nonlinear instability of an ideal fluid, AHP, 1997.

Difficulties:

1. Absence of the spectral mapping theorem: $\sigma(e^L) \neq e^{\sigma(L)}$ in general (typical for hyperbolic eqns)
2. EN is time reversible (no asymptotic stability), essential spectrum at the imaginary axis
3. Dependence of the final answer on the phase space (stable in H^1, unstable in L^2)

④ Thiner effects, like Landau-damping may occur.

Example of Sudovich (explaining how the stability may depend on the phase space)

$$\partial_t u - x\,\partial_x u = 0, \quad u|_{t=0} = \varphi$$

Explicit solution via characteristics

$$u(t,x) = \varphi(x e^{-t}) \Rightarrow \|u(t)\|_{L^p} = e^{-\frac{t}{p}} \|u(0)\|_{L^p}$$

$\Rightarrow$ Rate of convergence in L^p depends on p. If we modify slightly the equation

$$\partial_t u - x\,\partial_x u = \frac{1}{2} u$$

it will be stable for $1 \le p \le 2$ (exp. stable if $p<2$) and unstable for $p>2$.

Back to our topic - Radially symmetric vorticies:

Assumption: $G(x) = G(|x|)$ where $G(s) \sim s^{-\alpha}$ when $s \to \infty$ with $\alpha \in (0,2)$

Then, using BS-law, we get

$$\bar{u}(x) = \frac{x^{\perp}}{|x|^2} \int_0^{|x|} s\,G(s)\,ds, \quad x^{\perp} = (-x_2, x_1)$$

Remark: $\bar{u} \in L^\infty(\mathbb{R}^2)$ if $G \in L^2(\mathbb{R}^2)$.

indeed $|\bar{u}(x)| \le \frac{1}{|x|} \cdot \left(\int_0^{|x|} s\, G^2(s)\, ds \right)^{1/2} \cdot \left(\int_0^{|x|} s\, ds \right)^{1/2} \le \frac{1}{|x|} \|G\|_{L^2(\mathbb{R}^2)} \cdot (2\pi)^{1/2} |x|$

$\le C \|G\|_{L^2(\mathbb{R}^2)}$.

Let also. $\Omega(s) = \frac{1}{s^2} \int_0^s z\, G(z)\, dz$. Then $\bar{u}(x) = \Omega(|x|)\, x^{\perp}$

Let now $w(x_1, x_2)$ be a function from $L^2(\mathbb{R}^2)$ - scalar!

we write it in polar coordinates $w(x_1, x_2) = w(R, \varphi)$

and expand in F-series w.r.t φ.

$$(*) \quad w = \sum_{k \in \mathbb{Z}}{}' w_k(R)\, e^{ik\varphi}, \qquad w_k(|x|) = \frac{1}{2\pi} \int_0^{2\pi} w(|x|, \varphi)\, e^{-ik\varphi}\, d\varphi$$

By Parseval

$$\|w\|^2_{L^2(\mathbb{R}^2)} = \sum_{k=-\infty}^{\infty}{}' \int_0^\infty s\, |w_k(s)|^2\, ds$$

Aim: To write out more or less explicit formulas for operators L_a and L_c for radially symmetric vortex $G(|x|)$ in polar coordinates.

Laplace operator in polar coordinates:

$$\Delta = \partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2$$

$$\Rightarrow \quad \Delta\psi = \sum_{k\in\mathbb{Z}}{}' \Delta_k \psi_k(R)\, e^{ik\theta}, \qquad \Delta_k = \frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{k^2}{R^2}$$

Thus, to solve the equation $\Delta\psi = w$, we need to solve ODEs

$$\Delta_k \psi_k = w_k \qquad k\in\mathbb{Z}.$$

For $k=0$ we already have the formula for BS

$$BS(w)_0 = \frac{x^\perp}{|x|^2}\int_{|x|}^{s} \sigma\, w_0(\sigma)\, d\sigma$$

We need to consider the case $k\neq 0$ only

Lemma: Let $k\neq 0$. Then

$$\psi_k(s) = -\frac{1}{2|k|}\, s^{|k|}\int_s^\infty w_k(\tau)\tau^{1-|k|}\,d\tau - \frac{1}{2|k|}\, s^{-|k|}\int_0^s w_k(\tau)\tau^{1+|k|}\,d\tau$$

Proof: The ODE has two fundamental solutions $s^{|k|}$ and $s^{-|k|}$ + variation of constants f-la + proper boundary conditions.

Reminder $u(R,\varphi) = u\left(\sqrt{x_1^2+x_2^2}, \arctan\frac{x_2}{x_1}\right)$

so $\partial_{x_1} u = \partial_R u \cdot \frac{x_1}{R} + \partial_\varphi u \cdot \frac{-x_2}{R^2}$

$\partial_{x_2} u = \partial_R u \cdot \frac{x_2}{R} + \partial_\varphi u \, \frac{x_1}{R^2}$

$\nabla^\perp u = \partial_R u \, \frac{x^\perp}{R} - \partial_\varphi u \, \frac{x}{R^2}$

and if $u = \psi_k(R) e^{ik\varphi} \Rightarrow$

$$\nabla^\perp u = \left(\psi_k'(|x|) \frac{x^\perp}{|x|} - i k \psi_k(|x|) \frac{x}{|x|^2} \right) e^{ik\varphi}$$

$$(\nabla^\perp u, \nabla) G = i k \, \psi_k(|x|) \frac{G'(|x|)}{|x|}$$

Finally

$$(A_c w)_k = -\frac{i \operatorname{sgn} k}{2} s G'(s) \left(s^{|k|-2} \int_s^\infty \tau^{1-|k|} w_k(\tau) d\tau + s^{-2-|k|} \int_0^s \tau^{|k|-1} w_k(\tau) d\tau \right)$$

$$= \int_0^\infty Q_k(s,\tau) w_k(\tau) \tau \, d\tau$$

$$Q_k(s,\tau) = \frac{i}{2} \operatorname{sgn} k \begin{cases} s^{|k|-2} \tau^{-|k|} & s G'(s), \ \tau \geq s \\ s^{-|k|-2} \tau^{|k|} & s G'(s), \ \tau \leq s. \end{cases}$$

Zero component is equal to zero.

Remark We also have formulas for the components of BS-operator

$$BS(w)_k(x) = |x|^{|k|-2} \int_{|x|}^{\infty} w_k(\tau)\tau^{1-|k|}\,d\tau \left(-\frac{1}{2}x^{\perp} + \frac{i}{2}\operatorname{sgn}(k)x\right) +$$

$$+ |x|^{-|k|-2} \int_{0}^{|x|} w_k(\tau)\tau^{1+|k|}\,d\tau \left(\frac{1}{2}x^{\perp} + \frac{i}{2}\operatorname{sgn}(k)x\right)$$

We claim that: $\boxed{BS(w)_k(x) \in L^{\infty}(\mathbb{R}^2)}$ if $\boxed{k \neq \pm 1}$ and $\boxed{w \in L^2(\mathbb{R}^2)}$

Indeed $\left|\int_{|x|}^{\infty} w_k(\tau)\tau^{1-k}\,d\tau\right| \leq \left(\int_{|x|}^{\infty} \tau|w_k(\tau)|^2 d\tau\right)^{1/2} \cdot \left(\int_{|x|}^{\infty} \tau^{1-2|k|}\,d\tau\right)^{1/2} \leq$

$$\leq C\|w_k\|_{L^2(\mathbb{R}^2)} \cdot |x|^{1-|k|} \Rightarrow$$ the first term is globally bounded

the second term is estimated analogously

In contrast to this, for $k = \pm 1$, the first integral is ill posed in $L^2(\mathbb{R}^2)$.

$\Rightarrow$ if we start from $w \in L^2(\mathbb{R}^2)$, the ± 1st component $BS(w)_k(s)$ is not in L^{∞} (and even not in $\mathcal{D}'$), but should be defined by modulo of constants.

All other components are well defined!

This was surpicing for me! I expected problems with 0 component.

General theory: BS operator is a pseudo-differential operator with the symbol $i\,\xi^{\perp}/|\xi|^2$ – of order -1. It acts from $L^p \mapsto \dot{W}^{1,p}$ – homogeneous Sobolev space = completion of $\mathcal{D}(\mathbb{R}^2)$ wrt the norm $\|\nabla u\|_{L^p}$.

we know that $\dot{W}^{1,p}$ can be realized as a subset of $L^q(\mathbb{R}^2)$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$ if $q < \infty$. If $\frac{1}{p} - \frac{1}{2} \le 0$

it can be realized in $\mathcal{D}'/\text{consts}$ – not a usual function or distribution!

$p=2$ limit exponent where we need to factorize by the constants – may be a problem!

Standard solution: Take $w \in L^p(\mathbb{R}^2)$ with $p < 2$ OR even $w \in L^p(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)$ with $p_1 > 2$. Then $BS(w)$ – is well defined and, e.g. $\|BS(w)\|_{L^\infty} \le C\|w\|_{L^p \cap L^{p_1}}$

But then you lose the Hilbert structure! $\Rightarrow$ a lot of complications in the theory (convergence of F-series, energy type estimates, etc.)

Probably, nor of them are crucial, but this is not checked.

Alternative trick: just to drop out the ± 1 th modes

Consider the space $\boxed{w \in \tilde{L}_2(\mathbb{R}^2) = \{w \in L^2,\ w_{\pm 1}(s) \equiv 0\}}$

This is good for the linear theory, but may be not so good for the non-linear one (this property may be lost for products of functions). Will be not lost if we require that $w_k \equiv 0$ for all odd k. Then the non-linear theory will work as well (2 fold vector fields or more general m-fold vector fields with $m \geq 2$). Not- essential for us!

The following result is crucial for the perturbation theory.

Theorem: Under the above assumptions, the operator $\boxed{A_c w := (BS(w), \nabla) G}$ is compact as an operator in the space $\tilde{L}_2(\mathbb{R}^2)$.

Proof: Following Vishik, we compute the Hilbert-Schmidt norm of the integral operator with the kernel $Q_k(s,\tau)$ which defines the kth component of the BS operator

$$\| A_c(\cdot)_k \|^2_{HS} = \int_0^\infty \int_0^\infty |Q_k(s,\tau)|^2 s\, ds\, \tau\, d\tau$$

We use that $\boxed{G'(s) \leq C(1+s)^{-1-\alpha}}$

$$\|\cdot\|_{HS}^2 \le C \iint_{\tau\le s} s^{-2|k|-2}(1+s)^{-2-2d}\tau^{2|k|}\,s\,ds\,\tau\,d\tau +$$

$$+C\int_{\tau\ge s} s^{2|k|-2}(1+s)^{-2-2d}\tau^{-2|k|}\,s\,ds\,\tau\,d\tau \le$$

(integration wrt τ)

$$\le C\frac{1}{K}\int_0^\infty s^{-2|k|-2}\,s\,(1+s)^{-2-2d}s^{2|k|+2}\,ds +$$

$$C\frac{1}{K}\int_0^\infty s^{2|k|-2}\,s\,(1+s)^{-2-2d}s^{2-2|k|}\,ds \le$$

$$\frac{C}{|k|}\int_0^\infty s\,(1+s)^{-2-2d}\,ds \le \frac{C}{|k|}.$$

Finally, since the singular values of A_c is the union of singular values of $A(\cdot)_k$, we have that for $p>2$

$$\|A_c\|_{\mathfrak{S}_p}^p := \mathrm{TR}((TT^*)^{p/2}) = \sum_j\sum_k s_{j,k}^p \le \sum_k' \|A(\cdot)_k\|_{HS}^p \le C\sum_{k=2}^{\infty'}\frac{1}{K^{p/2}} < \infty$$

where we denote by $\{s_{j,k}\}_{j=1}^\infty$ - the singular values of $A_c(\cdot)_k$ and use that $\|x\|_{\ell_p} \le \|x\|_{\ell_2}$.

Thus, the operator $A_c \in \mathfrak{S}_p$ for $p>2$ (Shatten class) and by this reason is compact. □

Remark: There are many other ways to get the compactness, but this is the most elegant what I have seen.
Also the quantitative bounds for the compactness can be obtained ($\exists B(w) \in H^{\varepsilon}(\mathbb{R}^2)$ for some $\varepsilon>0$ + tail estimates)

Let us now return to the anti-symmetric part of the linearized operator $L = L_a + L_c$

$$\boxed{L_a w := -(\bar{v}, \nabla) w}$$

Sorry, I started with L and continued with A. Just identify L and A.

Since $\operatorname{div} \bar{v} = 0$, this operator is unbounded anti-symmetric and it is anti-self-adjoint with the domain $\mathcal{D}(L_a) = \{ w \in L_2(\mathbb{R}^2), \; L_a w \in L_2(\mathbb{R}^2) \}$

We have explisit formula for $\bar{v}$ through $G(|x|)$ so we may find (easily, no integral operators)

$$\boxed{L_a(\cdot)_k w_k = -ik\, \Omega(|x|)}, \qquad \boxed{\Omega(s) = \frac{1}{s^2} \int_0^s \tau\, G(\tau)\, d\tau}$$

Finally, we have found the H-space where $\boxed{L = L_a + L_c}$ is split into a sum of anti-symmetric and compact operators and reduce everything to an uncoupled system of equations for radial modes the "coefficients" are expessed in terms of vortex profile $G(|x|)$ to be found.