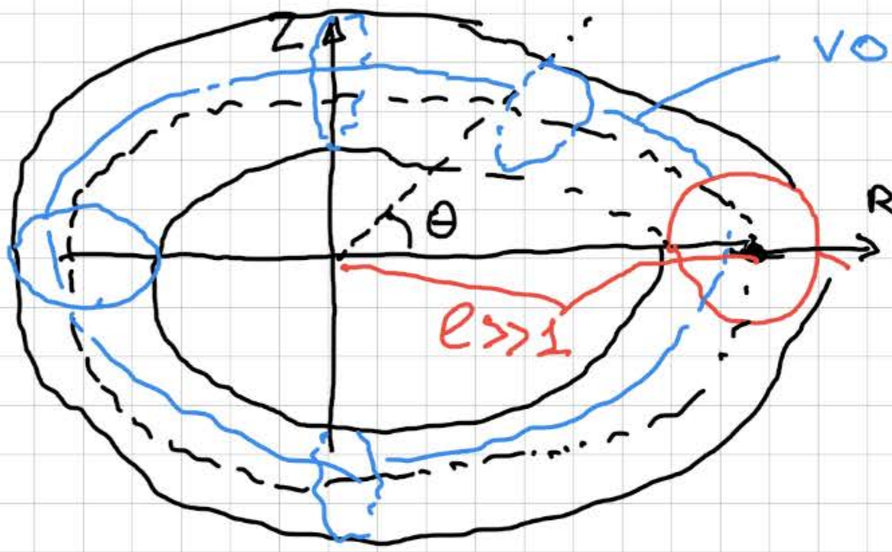
Lecture Instability in 3D case via big vortex rings

Aim: To construct unstable solutions of 3D Euler and Navier-Stokes equations

Idea: lift the 2D instability to 3D case



vortex ring profile

localized 2D vortex

$\ell \to \infty$ 2D Euler equation

+ perturbation arguments

nor a priory evident justification is needed!

Axisymmetric solutions in cylindrical coordinates:

$$u = u_R(R,Z)\,e_R + u_\varphi(R,Z)\,e_\varphi + u_Z(R,Z)\,e_Z$$

+ swirl free condition $u_\varphi \equiv 0$

$$u = u_R(R,Z)\,e_R + u_Z(R,Z)\,e_Z$$

Ansatz

PRESERVES under both Euler and Navier-Stokes flows!

Key feature: We arrive at 2D system for (u_R, u_z) which is close to the standard 2D Euler/Navier-Stokes equation (up to some extra difficulties related with singularity at $R=0$ - not essential for us). In particular, vorticity stretching term disappears and we have scalar vorticity as in 2D case:

$$\boxed{\vec{w} = \operatorname{curl} \vec{u} = -w\, e_\theta}$$

We start from Euler equations in vorticity form

$$\boxed{\partial_t w + u \cdot \nabla w - \frac{u_R}{R} w = 0}$$

$\nabla = (\partial_R, \partial_z)$

u is related with w by Biot-Savart law:

$$\boxed{\begin{gathered}\left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{1}{R^2} + \partial_z^2\right)\psi = w \\ u = -\partial_z \psi\, e_R + \left(\partial_R + \frac{1}{R}\right)\psi\, e_z\end{gathered}}$$

div-free condition reads

$$\boxed{\frac{1}{R}\frac{\partial}{\partial R}(R u_R) + \partial_z u_z = 0}$$

Construction: Let $(\tilde{u}_R, \tilde{u}_z)$ be the velocity profile which is $C_0^\infty(\mathbb{R}^2)$ and $(\tilde{u}_R, \tilde{u}_z) \equiv 0$ if $R^2+z^2 \le M^2$

Change $R \leftrightarrow R-\ell$, $\ell \gg R$ to centralize the vortex

Then, linearized vorticity eqn. reads:

$$\partial_t w = -L_\ell w, \quad L_\ell w := \tilde{u}\cdot\nabla w + u\cdot\nabla\tilde{w} - \frac{\tilde{u}_R}{\ell+R} w - \frac{u_R}{\ell+R}\tilde{w}$$

similar to 2D case

small as $\ell \to \infty$.

Biot-Savart law:

$$u = BS_\ell(w)$$

$$\begin{cases} \partial_R^2\psi + \frac{1}{R+\ell}\partial_R\psi - \frac{1}{(R+\ell)^2}\psi + \partial_z^2\psi = w \\ R \in [-\ell, \infty) \qquad \partial_z\psi|_{R=-\ell} = 0 \end{cases}$$

$$u = -\partial_z\psi\, e_R + \left(\partial_R + \frac{1}{R+\ell}\right)\psi\, e_z$$

again small

Problems: ① perturbation is not small near $R=-\ell$ ⇒ we need to fight with the "artificial" singularity at $R=-\ell$. ② If you take (u_R, u_z) 2D div-free it will be not 3D div-free: correction is needed!

The second problem (correcting divergence) is simple

need to find the corrector V_ε such that

$$\boxed{\operatorname{div}\left((R+\ell)(\tilde{u}-V_\varepsilon)\right)=0}$$ - usual $\operatorname{div}_{R,z}$

we know that

$$\operatorname{div}(\tilde{u})=0$$ since $\tilde{u}$ is the 2D profile

$$\Rightarrow \quad \boxed{\operatorname{div}((R+\ell)V_\varepsilon)=\tilde{u}_R}$$ $\tilde{u}$ is $C_0^\infty(B_R)$ + radially sym

$$\Rightarrow \boxed{\tilde{u}(x)=\frac{x^\perp}{|x|^2}\int_0^{|x|} s\,G(s)}, \quad G\in C_0^\infty$$ is the vorticity profile

= mean value of $\tilde{u}_R$ is ZERO $\Rightarrow$ Bogovski operator

$$\exists\, W\in C_0^\infty(B_R) \quad \operatorname{div} W=\tilde{u}_R \Rightarrow \boxed{V_\varepsilon=\frac{W}{R+\ell}\underset{\ell\to\infty}{\to}0 \text{ in } C_0^\infty}$$

$\Rightarrow$ We can replace $\tilde{u}$ by $\tilde{u}-V_\varepsilon$ up to small smooth correction - not dangerous terms

$\Rightarrow$ for simplicity we omit this correction in what follows.

First problem is a bit more interesting.

Key observation: Let w solves

$$\boxed{L_\varepsilon w = \lambda w} \text{ with } \lambda \neq 0 \Rightarrow \boxed{\operatorname{supp}(w) \subset B_M}$$

indeed since $\operatorname{supp}(\tilde{u}) \subset B_M \Rightarrow \operatorname{supp}(L_\varepsilon w) \subset B_M$ □.

$\Rightarrow$ we may cut-off w outside of $[-M, M]$ and consider L_ε in the space $L_2(-M, M)$ independent of ε! with the domain $\mathcal{D}(L_\varepsilon) = L_2(-M, M) \cap \{\tilde{u} \cdot \nabla w \in L^2\}$ depends on ε due to the corrector - not essential for us.

$\Rightarrow$ "small" terms in L_ε are really small and not essential.

But essential dependence on ε remains in the Biot-Savart law: here we need 1) restrict w to $[-M, M]$ (zero extension) 2) solve the equation

$$\partial_R^2 \psi + \frac{1}{R+\varepsilon} \partial_R \psi - \frac{\psi}{(R+\varepsilon)^2} = w, \quad \partial_n \psi|_{R=-\varepsilon} = 0 \text{ and find } u$$

3) Restrict it again to $[-M, M]$ cannot do cut off here! (since ψ is not localized).

We need to prove that

$$(*)\qquad \boxed{\| BS_\ell(w|_{B_M}) - BS_{2d}(w|_{B_M}) \|_{L^2(B_M)} = \underset{\ell\to\infty}{O(1)} \|w\|_{L^2(B_M)}}$$

most technical part: can be done by standard estimates, but better to return back to Euclider coordinates and use the f-las for BS-kernels and harmonic analysis (or do mixture of these methods). Will be not considered here!

$\Rightarrow$ if (*) is obtained, we may use our general theorem on perturbations of isolated eigenvalues and get the unstable eigenvalue for the 3D Euler ($\ell \gg 1$)

We proved the following result:

Theorem: There exists an axisymmetric swirl free smooth vortex ring with the isolated unstable eigenvalue. The corresponding eigenfunction is also axisymmetric and swirl free.

(2D $\to$ 3D lifting is completed)

Last step: To lift the instability from Euler to Navier-Stokes equation

(very similar to the 2D case done before)

Easier to argue in Eucidean coordinates:

$$\bar{U} = u_R(R,Z)\, e_R + u_Z(R,Z)\, e_Z$$

$$\bar{\Omega} = \text{curl}\, \bar{u} = \Omega_\theta\, e_\theta$$

- unstable vortex ring for Euler constructed before

vorticity - axisymmetric pure-swirl functions

$L^2_{aps}(\mathbb{R}^3)$ - the space of such vector fields

Linearized vorticity equation:

say in L^2_{aps}

$$\partial_t \Omega = L_{\bar{u}}\Omega = -\bar{U}\cdot\nabla\Omega - (BS_{3D}(\Omega),\nabla)\bar{\Omega} - \{(\bar{\Omega}\cdot\nabla)BS_{3D}(\Omega) - (\Omega,\nabla)\bar{U}\}$$

1) st term is anti-symmetric

2) nd term is compact

$$M\Omega = -(\bar{U},\nabla)\Omega$$

$$K\Omega = (BS_{3D}(\Omega),\nabla)\bar{\Omega}$$

Biot-Savart operator in 3D is better that in 2D ($\dot{H}^1 \hookrightarrow L^6$) - compactness is immediate!

3rd term is new! It is not compact

$$\mathcal{B}\Omega = -(\Omega,\nabla)\bar{U} - (\bar{\Omega},\nabla)\,BS_{3D}(\Omega)$$

(this is the part which persists in vorticity stretching term)

Direct calculation:

$$\boxed{(\mathcal{B}\Omega)_\theta = \frac{1}{R}\bar{\Omega}_\theta U_R + \frac{1}{R}\Omega_\theta \bar{U}_R}$$

If the ring is big enough ($\varrho >> 1$)
the operator $\mathcal{B}$ is small.

$\Rightarrow$ for the perturbed eigenvalue problem:

$$\boxed{\varepsilon\left(\left(1+\frac{x}{2}\cdot\nabla\right)\Omega + \Delta\Omega\right) + M\Omega + K\Omega + \mathcal{B}\Omega = \mu\Omega}$$

we have exactly the same structure as in 2D.

$\Rightarrow$ isolated eigenvalue persists for small ε $\Rightarrow$ instability for the scaled NS problem in 3D.

Theorem 2: There exists a vortex ring equilibrium for the forced scaled 3D NS equation which is exponentially unstable.

(long way of constructing such an equilibrium is completed!)