Lecture: Completion of the non-uniqueness proof for 3D NS problem.

We have constructed a "big" vortex ring $U(x) \in C_0^\infty(\mathbb{R}^3)$ which possesses an exponentially unstable eigenvalue

Scaled NS equations: linearization

$$(*) \quad \begin{cases} \frac{1}{2}(1 + x\cdot\nabla)V + \Delta V - \mathbb{P}[(U\cdot\nabla)V - (V,\nabla)U] = \lambda V \\ \operatorname{div} V = 0 \end{cases} \qquad \operatorname{Re}\lambda >> 1$$

Being pedantic: we did this for the vorticity eqn. $V \in \dot{H}^1(\mathbb{R}^3)$ - homogeneous space

so we know only that

But we want to have $V \in L^2(\mathbb{R}^3)$ - ?

We have $w = \operatorname{curl} V \in L^2$ $\Rightarrow$ B-S law $V \in \dot{H}^1 \hookrightarrow L^6$

How to get $V \in L^2$?

1) $(U,\nabla)V \in L^2$ since $U \in C_0^\infty$ and $\nabla V \in L^2$

$(V,\nabla)U \in L^2$ since $V \in L^6$ and $U \in C_0^\infty$

$\Rightarrow H := \mathbb{P}[(U,\nabla)V - (V,\nabla)U] \in L^2(\mathbb{R}^2)$

2) Rewrite (*) as follows

$$\left(\tfrac{1}{2}+\Delta+\tfrac{1}{2}x\cdot\nabla-\lambda\right)V=H\in L^2$$

$$V\in\dot{H}^1$$

$\Rightarrow \exists!$ solution $\overline{V}\in L^2$ for this eqn; also $\overline{V}\in H^1$.

$$W=V-\overline{V}\in\dot{H}^1\subset L^6$$

$$\left(\tfrac{1}{2}+\Delta+\tfrac{1}{2}x\cdot\nabla-\lambda\right)W=0$$

Taking the curl, we get

(**) $$\left(\frac{3}{2}+\Delta+\frac{1}{2}x\cdot\nabla-\lambda\right)\Omega=0,\quad \Omega=\operatorname{curl}W\in L^2$$

3) Since λ is Big, the operator is invertible in L^2

$\Rightarrow \Omega=0\Rightarrow V=\overline{V}\in L^2.\Rightarrow$ $V\in H^s(\mathbb{R}^n)\ \forall s>0$

Why am I so pedantic? It looks obvious!

Reason: the operator $\frac{1}{2}x\cdot\nabla$ - is not so simple as you may expect! Yudovich example!!!

$\partial_t u - x\cdot\nabla u=0$ in L^2 x u $\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\frac{n}{2}\|u\|_{L^2}^2=0$

stable in $L^2(\mathbb{R}^n)$ for all $n\in\mathbb{N}$.

Same equation, but in H^1 may be not stable!

$$((X\cdot\nabla)u,\Delta u) = \left(1-\frac{n}{2}\right)\|\nabla u\|_{L^2}^2$$

for $n=1$: $(x\partial_x u, \partial_x^2 u) = \pm\frac{1}{2}(x,\partial_x(u^2)) = -\frac{1}{2}\|\partial_x u\|_{L^2}^2$

$$\Rightarrow \frac{1}{2}\partial_t\|\nabla u\|_{L^2}^2 + \left(\frac{n}{2}-1\right)\|\nabla u\|_{L^2}^2 = 0$$

$\Rightarrow$ different behavior in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$!!
(becomes unstable for $n=1$ and neutral for $n=2$)

The presence of the Laplacian cures this problem:
the operator $X\cdot\nabla$ is still not subbordinated to Δ
and $-\Delta - X\cdot\nabla$ is not a sectorial operator $\Rightarrow$
analytic semigroups are not helpful here.
But the extra terms related with (1) or (3/2)
are subbordinated $\Rightarrow$ most part of problems
disappear and everything is indeed "obvious"
much more difficult for the Euler equation
(not considered here)

Step 2: Linear semigroup $S_0(t)$ generated
by the equation

$$(L_0)\quad \begin{cases} \partial_t W = \left(\Delta + \frac{1}{2}(1+X\cdot\nabla)\right)W \\ W|_{t=0} = W_0 \end{cases} \quad \text{in } L^2 \text{ and } H^1$$

Lemma 1: The spectral radius in Both L_2 and H^1

$$\boxed{r(S_0(t)) \le e^{-1/4\, t}}$$

Proof: $\times (L_0)$ By w $\Rightarrow$ $\boxed{\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \frac{1}{4}\|w\|_{L^2}^2 = 0}$

$$\Rightarrow \boxed{\|w(t)\|_{L^2}^2 + \int_0^t e^{-\frac{t-s}{2}} \|\nabla w(s)\|_{L^2}^2 ds = e^{-t/2}\|w_0\|^2} \Rightarrow$$

$$\|S_0(t) w_0\|_{L^2} \le e^{t/4}\|w_0\|_{L^2} + \text{Gelfand's formula}$$

$\Rightarrow$ desired result in L^2.

Estimate in H^1: smoothing property $\times t\Delta w$

$$\boxed{\|w(t+1)\|_{H^1} \le C\|w(t)\|_{L^2}}$$

$\Rightarrow$ analogous estimate for the decay in H^1 $\square$

Step 2: perturbed linear equation:

$$(L)\quad \boxed{\begin{aligned} &\partial_t w = \left(\Delta + \tfrac{1}{2}(1 + x\cdot\nabla)\right) w - P\left[(v,\nabla)w + (w,\nabla)v\right] := L_0 w + Kw \\ &\operatorname{div} w = 0 \end{aligned}}$$

$w|_{t=0} = w_0$ in H_σ and H^1_σ

Denote the solution semigroup of (L) By $S(t)$

Lemma 2: The difference $S(t) - S_0(t)$ is a compact operator for every fixed $t > 0$.

Proof (for $t=1$) 1st: we multiply (L) by w and get the energy estimate:

$$\|w\|_{C(0,1;H^1_\sigma)\cap L^2(0,1;H^1_0)} \le C\|w_0\|_{L^2}$$

2nd: The function $z(t) = w(t) - S_0(t)w_0$ solves

$$(P) \quad \partial_t z = L_0 z + Kw, \quad z|_{t=0} = 0, \quad \operatorname{div} z = 0$$

since $U \in C_0^\infty \Rightarrow \|Kw\|_{L^2} \le C\|w\|_{H^1}$

$\Rightarrow \times (P)$ by Δw and using energy estimate

$$\|z(t)\|_{H^1} \le C\|w_0\|_{L^2}$$ - smoothing effect.

Still not enough for compactness - need tail estimates

since $U \in C_0^\infty \Rightarrow \|(U,\nabla)w + (w,\nabla)U\|_{L^2_\Sigma} \le C\|w\|_{H^1}$

$\Sigma(x) = (1+\varepsilon|x|^2)$ - Muckenhoupt weight in $\mathbb{R}^3$

$\varepsilon \ll 1$

$$\|Kw\|_{L^2_\Sigma} \le C_1\|w\|_{H^1}$$

Multiplying equation for z by ϱz

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2_\varrho} + \|\nabla z\|^2_{L^2_\varrho} + 2(\nabla z, \nabla \varrho z) + \frac{1}{4}(\mathrm{div}(x\varrho), z^2) - \frac{1}{2}(\varrho, z^2) = (Kw, \varrho z)$$

$|x \cdot \nabla \varrho| \sim \varepsilon x^2$, $|\nabla \varrho| \le \varepsilon \varrho \Rightarrow$

$$\frac{d}{dt}\|z\|^2_{L^2_\varrho} + \frac{1}{4}\|z\|^2_{L^2_\varrho} \le C\|w(t)\|^2_{H^1} \qquad t \le 1$$

+ energy estimate for $w \Rightarrow \|z(t)\|^2_{L^2_\varrho} \le C\|w_0\|^2_{L^2}$

since $H^1 \cap L^2_\varrho \Subset L^2$ $\quad S(t) - S_0(t)$ is compact in L^2

Compactness in H^1 is analogous $\square$

Corollary: $r_{ess}(S(1)) = r_{ess}(S_0(1)) \le e^{-1/4}$ both in H_b and H^1_b

and the semigroup $S(t)$ is exponentially unstable (isolated eigenvalues of finite algebraic multiplicity)

$\Rightarrow S(1)$ satisfies the relaxed hyperbolicity assumptions

Linear analysis is completed!

Non-linear equation - unstable manifold

(N) $\partial_t w = L_0 w + Kw - P[(w,\nabla)w]$, $\operatorname{div} w = 0$, $w|_{t=0} = w_0$

still 3D Navier-Stokes - problems with global smooth solutions + H^1_σ is a bad phase space (equation will remain supercritical $\Rightarrow$ bad diff properties even for local solutions)

Reasonable choice: $\boxed{\Phi = H^1_\sigma(\mathbb{R}^3)}$ problems with global solutions but we only need local ones

$\times$ (N) by Δu + key estimate

$$|((w,\nabla)w, \Delta w)| \le \|\nabla w\|^3_{L^3} \le \|\nabla w\|^{3/2}_{L^2} \cdot \|\Delta w\|^{3/2}_{L^2} \le$$
$$\le \frac{1}{2}\|\Delta w\|^2_{L^2} + C\|\nabla w\|^6$$

$\Rightarrow$ Energy type estimate for $Y(t) := \|w(t)\|^2_{H^1}$

$$\boxed{\frac{d}{dt} Y(t) \le C_1 Y(t) + C_2 Y^3(t)} \Rightarrow \text{existence of local solutions}$$

+ lifespan$(w) \to \infty$ as $\|w_0\|_{H^1} \to 0$.

$\Rightarrow$ solution semigroup is defined locally near zero $S(t)$ - exists for $w_0 \in B_\varepsilon(0)$ in H^1-metric + + it is C^∞-smooth w.r.t. initial data - straightforward calculations - no surpices!

Finally, all conditions for the abstract theorem on the existence of an essentially unstable manifold are verified.

$\Rightarrow$ $\exists$ a backward solution $w_a(t)$ of (N) such that

$$w_a(t) = e^{at}\eta(x) + e^{2at}\,o(1) \quad \text{as } t \to -\infty$$

$\mathrm{Re}\, a > 0$ - eigenvalue

$\eta = \eta(x)$ - eigenvector

OR which is the same, we have an equilibrium $U(\xi)$ for the scaled equation

$$(S)\quad \begin{cases} \partial_\tau u + \nabla p - \frac{1}{2}(1 + \xi\cdot\nabla_\xi)u - \Delta_\xi u + (u,\nabla_\xi)u = F(\xi) \\ \mathrm{div}_\xi\, u = 0 \end{cases}$$

as well as another solution $u(\tau,\xi)$ belonging to the exponentially unstable set of $U(\xi)$ + everything is smooth at least for $\tau \le 0$.

Return to non-scaled variables

$$u(t,x) = \frac{1}{\sqrt{t}}\,u(\tau,\xi), \quad \tau = \ln t, \quad \xi = \frac{x}{\sqrt{t}}$$

$$f(t,x) = \frac{1}{t^{3/2}}\,F(\tau,\xi)$$

$(-\infty, 0]$ in τ $\Leftrightarrow$ $[0,1]$ in t

instability is scaled var $\Rightarrow$ non-uniqueness in initial var

Final discussion:

(1) Why this is a Leray-Hopf solution

Equilibrium $U(\xi)$ $\rightarrow$ $u(t,x) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$, $u \in C_0^\infty$

self-similar solution

Regularity class

$u \in L^\infty(0,1; L^3)$ - exactly the critical space

singularity at $t=0$ only, but $u \in C([0,1], L^2)$

$u(0)=0$ easy to show

$\Rightarrow$ Leray-Hopf. Second solution $u_1(t)$ has the structure $u(t)$ + "something more regular"

— also Leray-Hopf.

(2) Why this is not a solution of the mill. problem?

The external force is NOT smooth at $t=0$!

$f \in L^1(0,1; L^2)$ $\Rightarrow$ energy solutions are well-defined

But the singularity is not generated by the equation. It is inserted artificially via the external forces. Moreover, such scenario cannot be realized in the unforced equation:

$u \in L^\infty(-\infty, 1; L^3)$ $\Rightarrow$ u is smooth

(Seregin et al.)

③ This has many interesting applications

ⓐ Straightforward one : similar result for Brinkman-Forchheimer equation

$$\partial_t u + (u,\nabla)u + \nabla p + \varepsilon u|u|^2 = \Delta u + f, \quad \operatorname{div} u = 0$$

The same scaling - similar scaled equation

if $\varepsilon << 1$ - small perturbation (regular) of the spectral problem → non-uniqueness.

if ε is big enough → uniqueness of a weak solution.

ⓑ More advanced : gives an alternative way to construct unstable solutions for N-S equations suitable to the case of whole space and b-d domains (to Kolmogorov's flows which work for periodic domains only)

opens the way to get the lower bounds for the fractal dimension of the corresponding global attractor.

END of the Spring Session !!