Lecture. Essentially unstable manifolds

Already done: an equilibrium $u_0(x)$ for the forced scaled 3D Navier-Stokes equation which is exponentially unstable is constructed.

Remains: To construct an "essentially" unstable manifold of this equilibrium

Then, returning back to non-scaled variables, we get the desired non-uniqueness.

unstable manifolds: standard theory

$S(t): \Phi \to \Phi$, Φ - phase space (B-space)

$S(t)$ - is a semigroup, 0 is an equilibrium

1-map $S = S(1)$, Assume that

$S \in C^{1+\alpha}(\Phi, \Phi)$, $\alpha > 0$, $S = S_0 + B$,

$S_0 \in \mathcal{L}(\Phi, \Phi)$, $B(0) = B'(0) = 0$

Linearization

small correction (non-linear)

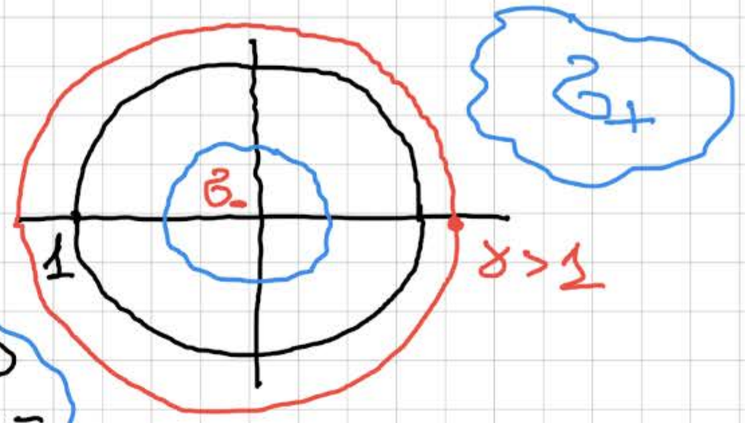
Hyperbolicity assumption:

$$\mathcal{C}(S_0) = \mathcal{C}_+ \sqcup \mathcal{C}_-$$

$$\mathcal{C}_+ \cap \overline{B}_1 = \emptyset$$
$$\mathcal{C}_- \subset B_1$$

$$\Rightarrow \quad \Phi = \Phi_{-+} \oplus \Phi_{--}$$

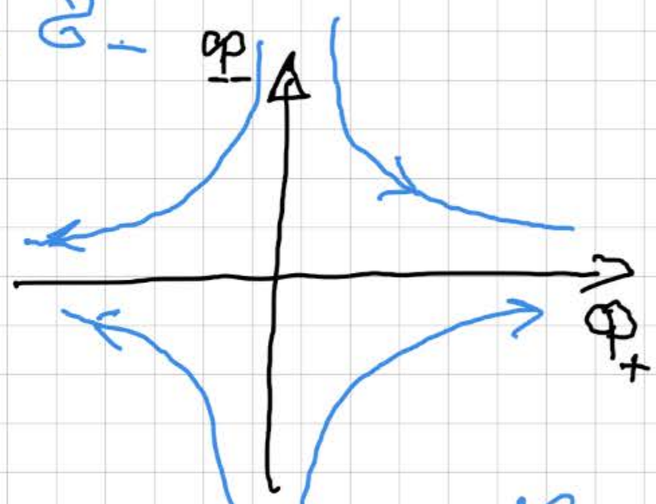
$$\exists\, \Phi_{\pm\pm} \subset \Phi_{\pm\pm}$$



spectral subspaces which correspond to $\mathcal{C}_+$ and $\mathcal{C}_-$

Exp. dichotomy:

$$\|S_0^n u_0\|_\Phi \le \gamma^{-n} \|u_0\|_\Phi, \quad u_0 \in \Phi_{--}$$
$$\|S_0^{-n} u_0\|_\Phi \le \gamma^{n} \|u_0\|_\Phi, \quad u_0 \in \Phi_{+-}$$



Linear saddle

$u_0 \in \Phi_{+-} \Leftrightarrow S_0^n u_0 \to 0$ (exp) as $n \to -\infty$

$u_0 \in \Phi_{--} \Leftrightarrow S_0^n u_0 \to 0$ (exp) as $n \to +\infty$

$\Phi_{+-} \Leftrightarrow$ solutions of the equation

$u_0 \in \Phi_{-+}$

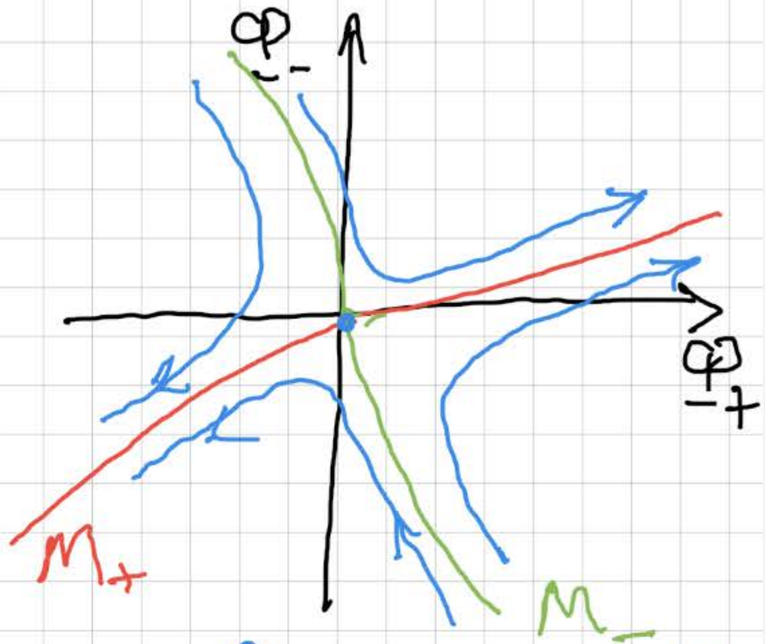
$$u_n = S_0 u_{n-1}, \quad n \le 0$$

$$u = \{u_n\}_{n=0}^{-\infty}, \quad u \in L_\gamma(\mathbb{Z}_-, \Phi)$$

$$\|u_\cdot\|_{L_\gamma} = \sup_{n \in \mathbb{Z}_-} \{\gamma^{-n} \|u_n\|_\Phi\}$$

Key observ for the PERRON method!

Standard fact: exp. dichotomies survive under small perturbations (will be just slightly deformed, but not destroed) $\Rightarrow$ stable/unstable manifolds ($\mathcal{M}_-$, $\mathcal{M}_+$) will be just perturbed analogues of $\underline{\Phi}_-$, $\underline{\Phi}_+$.



non-linear saddle

How to construct $\mathcal{M}_+$?

By solving the equation:

$$(*)\quad \begin{cases} u_n - S_0 u_{n-1} = B(u_{n-1}), \quad \Pi_+ u_0 = p \in \underline{\Phi}_+ \\ n \le 0 \qquad u \in L_\gamma(\mathbb{Z}_-, \underline{\Phi}) \end{cases}$$

small perturbation

by Implicit function theorem or BCT.

If (*) is solved, then the graph of $\mathcal{M}_+$ is given by the function:

$$M_+ : \underline{\Phi}_+ \to \underline{\Phi}_-, \qquad M_+ \underline{p} := \Pi_- u_0(p)$$

$$\mathcal{M}_+ = \{ p + M_+(p), \; p \in B_\varepsilon(0, \underline{\Phi}_+) \}$$

local unstable manifold (uniqueness, invariance)

stable - analogously

How to solve (*)?

Lemma: under above assumptions on the spectrum $\mathcal{G}(S_0)$, the problem

$$u_n - S_0 u_{n-1} = h_{n-1},\ n \le 0, \quad \Pi_+ u_0 = p$$

is uniquely solvable in $L_\gamma(\mathbb{Z}_-, \underline{\mathcal{P}})$ for any $p \in \underline{\mathcal{P}}_+$ and $h \in L_\gamma(\mathbb{Z}_-, \underline{\mathcal{P}})$

$$u = \overline{\Pi}_\gamma h + S_+ p$$

where $\overline{\Pi}_\gamma$ and S_+ are bounded linear operators

Proof: Explicit formula for the solution + Gelfand's formula for the spectral radius

Solving (*): We rewrite it in the form of

$$u = T_\gamma B(u) + S_+ p \qquad (**)$$

in a small neighbourhood of 0 B is small together with its 1st derivative $\Rightarrow$ Implicit function theorem or BCT work $\Rightarrow$ $\exists!$ of the desired solution $\Rightarrow$ $\mathcal{M}_+$ is constructed

Important! We have constructed $\mathcal{M}_+$ for the discrete time $n \in \mathbb{Z}$ and for 1-map $S = S(1)$ and its iterations, but it will be automatically the unstable manifold for the continuous time and the semigroup $S(t)$, $t \geq 0$. No extra actions is needed! This follows from the fact that

discrete trajectories convergent to zero $\Longleftrightarrow$ continuous trajectories convergent to zero

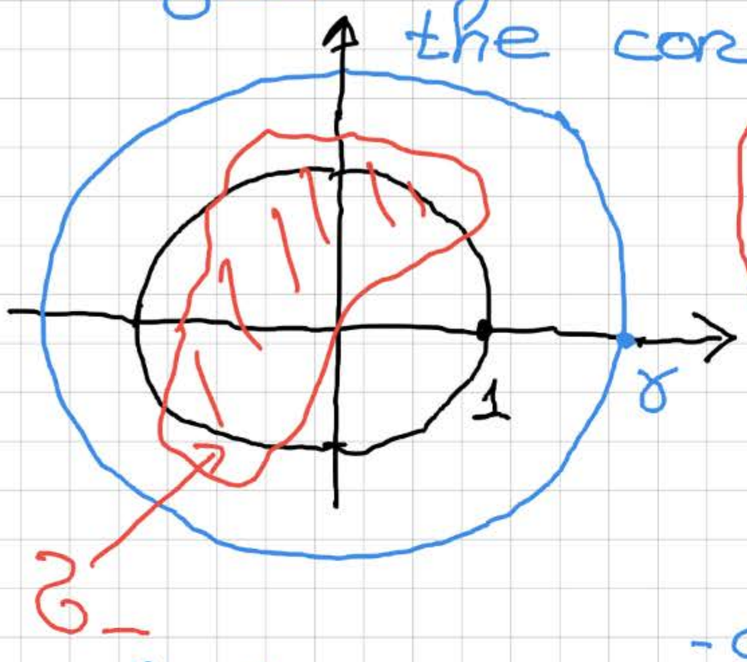
(at least if the map $u_0 \to S(t)u_0$ is locally continuous near 0 for every fixed t)

Advantages of the discrete approach:

1. We are working with bounded operators only! need not to consider generators, analytic semigroup, etc.
2. All conditions are formulated in terms of the map $S = S(1)$ no resolvents estimates, spectral mapping theorems are involved.
3. Much more natural construction - uses only what is really necessary.

Advise: Do not read/study the proofs used analytic semigroups!!

Way I: Preserve spectral gap, but remove the condition $\boxed{\sigma(S_0) \cap \{|z|=1\} = \emptyset}$



$$\underline{\Phi} = \underline{\Phi}_+ \oplus \underline{\Phi}_-$$

No changes with construction

$\mathcal{M}_+$ - usually referred as an essentially unstable manifold

- consists of trajectories decaying fast enough (like γ^{-n}) as $n \to -\infty$

($\mathcal{M}_+$ - is smooth (C^∞) and unique if S is smooth)

Exactly the same construction and exactly the same arguments.

$\mathcal{M}_-$ (which is called the center-stable manifold also exists, but may be not-unique, requires the cut-off procedure and may be not C^∞ even if everything is smooth (similarly to the center manifold theory)

We are not interested in $\mathcal{M}_-$ here, only in $\mathcal{M}_+$

Corollary: Let $\sigma_{ess}(S_0) \subset B_\gamma(0)$, but $\sigma(S_0) \not\subset \overline{B_\gamma(0)}$

Then there exists a finite-dimensional essentially unstable manifold $\mathcal{M}_+$.

Proof: Since outside of $B_\gamma(0)$ we have only isolated eigenvalues of finite multiplicity, we may change γ (increase slightly) to have $\sigma(S_0) \cap \{|z|=\gamma\} = \emptyset$ and the spectral space $\mathcal{P}_+$ which corresponds to $\sigma_+ = \sigma(S_0) \cap \{|z|>\gamma\}$ is finite-dimensional and not trivial.

$\Rightarrow$ $\mathcal{M}_+$ as a graph or $\mathcal{P}_+$. This manifold is unique and is invariant and coincides with the analogous manifold for the continuous case. □

Remark: This corollary perfectly fits to our purposes (for the scaled 3D Navier-Stokes)

But before we consider one more possibility to relax hyperbolicity assumption and get rid of a spectral gap.

Way II: Relaxing the spectral gap assumption

S.Z. "Multi-parameter semigroups and attractors of reaction-diffusion equations in $\mathbb{R}^n$"
Trans. Moscow Math. Soc., 2004, 105–160.

Key observation: if $u \in L_\gamma(\mathbb{Z}_-, \Phi)$ and $B(0) = B'(0) = 0$ and $B \in C^{1+\alpha}(\mathbb{Z}_-, \Phi)$ then $B(u) \in L_\gamma^{1+\alpha}(\mathbb{Z}_-, \Phi)$

so we may try to solve

$$(**) \quad u_n - S_0 u_{n-1} = B(u_{n-1})$$

not in $L_\gamma(\mathbb{Z}_-, \Phi)$ but in $L_\gamma^{1+\alpha}(\mathbb{Z}_-, \Phi)$

For instance, if $|\mathcal{B}(S_0)| < \gamma^{1+\alpha}$ this problem is solvable

Realization leads to

Theorem: Let $S(u) = S_0 u + B(u)$, $B \in C^{1+\alpha}(\Phi, \Phi)$, $B(0) = B'(0)$ and let $\Phi_+ \subset \Phi$ be closed and invariant w.r.t S_0. Assume

$$\inf |\mathcal{B}(S_0|_{\Phi_+})| > \gamma > 1 \quad \text{and} \quad \gamma^{1+\alpha} > |\mathcal{B}(S_0)|$$

Then there exists $C^{1+\alpha}$-map $V: B_\varepsilon(0, \Phi_+) \to L_\gamma(\mathbb{Z}_-, \Phi)$ s.t.

$$\| V_n(V_+) - (S_0|_{\Phi_+})^n V_+ \|_\Phi \le C \gamma^{(1+\alpha)n} \|V_+\|_\Phi^{1+\alpha}$$

$$\forall\, V_+ \in B_\varepsilon(0, \Phi_+)$$

$V_n(V_+)$ – solves $(**)$.

Idea of the proof: Denote by $S_+ V_+ \in L_\gamma(Z_-, \underline{\Phi})$ the solution of $V_n = S_0 V_{n-1}$, $V_0 = V_+ \Rightarrow V_n = (S_+)_n V_+ = S_0^n V_+$ and introduce $W_n = U_n - V_n$ which solves

$$W_n - S_0 W_n = B(W_{n-1} + V_{n-1}) \qquad B(W + S_+ V) \in L_{\gamma^{1+d}}(Z_-, \underline{\Phi})$$

equation is invertible in this space, no initial conditions are needed since all of the spectrum is in $B_{\gamma^{1+d}}(0)$, so we have

$$\boxed{W = T_{\gamma^{1+d}} B(W + S_+ V_+)} \quad \text{in } L_{\gamma^{1+d}}(Z_-, \Phi)$$

solving this equation by Implicit Function theorem we get the desired result. □

where is the unstable manifold?

$$\boxed{\mathcal{M}_+ := \{ V_0(V_+),\ V_+ \in B_\varepsilon(0, \underline{\Phi}_+) \}}$$

Indeed: $V_0 : B_\varepsilon(0, \underline{\Phi}_+) \to \underline{\Phi}$ is smooth injective map with

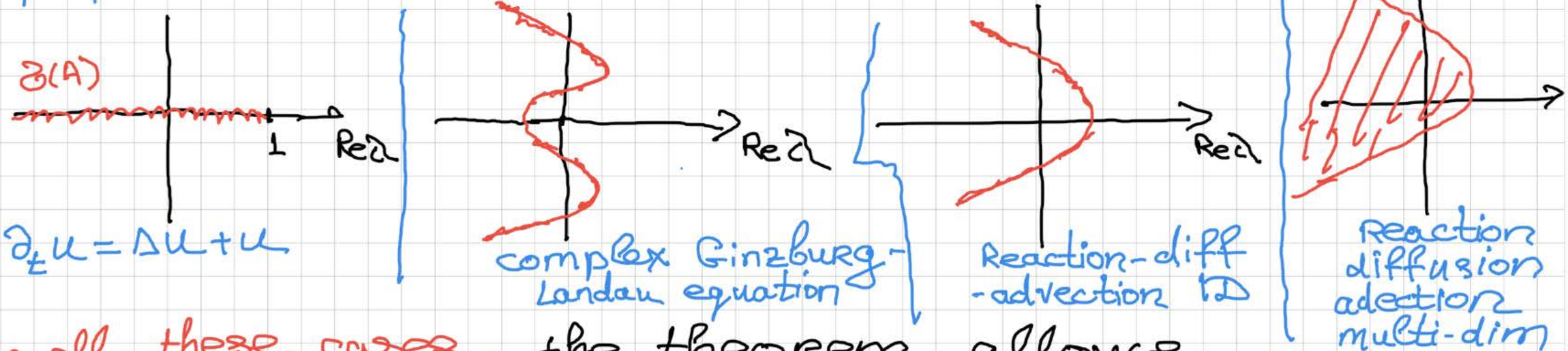
$$\boxed{\| V_0(V_+) - V_+ \|_{\underline{\Phi}} \le C \| V_+ \|_{\underline{\Phi}}^{1+d}} \Rightarrow \text{Tangent plane to } \mathcal{M}_+ \text{ at } 0 \text{ is } \underline{\Phi}_+$$

If $\underline{\Phi}_+$ is **complementable** in $\underline{\Phi}$ $\Rightarrow$ $\mathcal{M}_+$ can be written as a graph over $\underline{\Phi}_+$

Very general result: $\mathcal{P}_+$ may be finite or infinite-di
even that $\mathcal{P}_+$ is complementable is not necessary,
spectral gap is not necessary, etc.

Key problem: to find the proper invariant subspace $\mathcal{P}_+$
(may be difficult/impossible in general – case study)
often can be found by Fourier transform or spectral
theorem - e.g. if the generator is self-adjoint.

Remark: The absence of spectral gaps is typical
for PDEs in unbounded domains:



In all these cases, the theorem allows
us to construct massive essentially unstable
manifolds. Has been invented for obtaining
the lower bounds for the Kolmogorov's entropy
of global attractors.