Lecture 5: Steady Stokes problem.

Non-linear localization (continued)

Previous lecture: interior estimate + boundary estimates in tangential direction:

(1) $$\boxed{(\varphi \Delta u, \Delta u) + (\varphi f'(u)\nabla u, \nabla u) \leq C(1 + \|g\|_{L^2}^2)}$$

(interior)

(2) $$\boxed{\|\nabla \partial_x u\|_{L^2}^2 + (|u|^{R-1}, |\partial_x u|^2) \leq C(1 + \|g\|_{L^2}^2)}$$

(boundary tangential)

+ some extra pressure estimates like

$$\boxed{\|\partial_x p\|_{L^2 + W^{1,q}} \leq C(1 + \|g\|_{L^2})}$$

Aim for today: complete the normal direction estimate using the equation:

(**) $$\begin{cases} -\Delta u + f(u) + \nabla p = g \\ u|_{\partial\Omega} = 0, \ \operatorname{div} u = 0 \end{cases}$$

Does not work similarly to the linear case!

Do not know why exactly? May be that we are missing something??

In the case $f \equiv 0$: (1) We have the L^2-control of $\partial_\tau p$, not $(L^2 + W^{1,q})$-one. $W^{1,q} \hookrightarrow L^2$ only if $R \le 5$.

(so, the estimate is weaker in nonlinear case)

Then, we write the eqn for $u_\tau := u \cdot \tau$

(2) $$-\Delta u_\tau + f(u) \cdot \tau + \partial_\tau p = g \cdot \tau + L.O.T$$

and get the estimate for $\partial_n^2 u_\tau$ in the L^2-norm from here.

(3) from $\operatorname{div} u = 0 \Rightarrow \partial_n u_n + \partial_{\tau_1} u_{\tau_1} + \partial_{\tau_2} u_{\tau_2} = L.O.T$

+ diff $\partial_n (\operatorname{div} u)$ gives us estimate for $\partial_n^2 u_n$ and completes the estimate for H^2-norm of u

(4) Finally, we find $\partial_n p$ from equation on u_n and complete the max-reg est.

Non-linear case: we use different (weaker) scheme: (A) $\times$ (**) By $-\Delta u$ we get:

$$\|\Delta u\|_{L^2}^2 + (|u|^{R-1}, |\nabla u|^2) \le C(1 + \|g\|_{L^2}^2) + C\|\nabla p\|_{L^2}^2$$

need to estimate!

(B) write the equation in the form

$$\begin{cases} -\Delta u + \nabla p = -f(u) + g \\ \operatorname{div} u = 0 \end{cases}$$

and apply the linear estimate

$$\|\nabla p\|_{L^2}^2 \le C(1+\|g\|_{L^2}^2) + C\|f(u)\|_{L^2}^2 \qquad (\star)$$

$$\Rightarrow \boxed{\|\Delta u\|_{L^2}^2 + (|u|^{R-1}, |\nabla u|^2) \le C(1+\|g\|_{L^2}^2) + C\|f(u)\|_{L^2}^2}$$

need to estimate $\|f(u)\|_{L^2}^2 \sim \|u\|_{L^{2R}}^{2R}$.

How? Using tangential regularity + anisotropic Sobolev embeddings!

Namely: $(|u|^{R-1}|\nabla u|^2) \sim (\nabla(|u|^{\frac{R+1}{2}}), \nabla(|u|^{\frac{R+1}{2}}))$

Sobolev embedding $H^1 \hookrightarrow L^6$

$$\|u\|_{L^{3(R+1)}}^{R+1} \le C(|u|^{R-1}, |\nabla u|^2)$$ - not enough!

Anisotropic version:

$$\|v\|_{L^6} \le C\|\partial_{x_1} v\|_{L^2}^{1/3} \cdot \|\partial_{x_2} v\|_{L^2}^{1/3} \cdot \|\partial_{x_3} v\|_{L^2}^{1/3}$$

$$\Rightarrow \boxed{\|f(u)\|_{L^{3q}}^{q} \sim \|u\|_{L^{3(R+1)}}^{R+1} \le C(|u|^{R-1}, |\partial_2 u|^2)^{1/3} \cdot (|u|^{R-1}, |\partial_2 u|^2)^{2/3}}$$

already estimated!

$$\Rightarrow \quad \boxed{\|f(u)\|_{L^{\frac{3}{2}}}^{q} \leq C\left(1+\|g\|_{L^2}^2\right)^{2/3}\left(|u|^{R-1}, |\nabla u|^2\right)^{1/3}} \qquad (v)$$

and, analogously, using the controls $\|\nabla(\partial_t u)\|_{L^2}^2$ and $\|u\|_{W^{3,q}}$ $\Rightarrow$

$$\boxed{\|u\|_{W^{1,3q}}^2 \leq C\left(1+\|g\|_{L^2}^2\right)}$$

comes from energy estimate

Corollary (main estimate):

$$\boxed{\|u\|_{H^2}^2 + \|\nabla p\|_{L^2}^2 + \|f(u)\|_{L^2}^2 \leq C\left(1+\|g\|_{L^q}\right)^{\beta}\left(1+\|g\|_{L^2}\right)^{3}} \qquad (m)$$

$$\beta = \max\left\{2, \frac{4(2-q)}{3q-2}\right\}$$

Important! : $\beta < 4$

Idea of the proof: Do interpolation

$$\boxed{\|f(u)\|_{L^2}^2 \leq \|f(u)\|_{L^q}^{\frac{3q-2}{2}} \cdot \|f(u)\|_{L^{3q}}^{\frac{3(2-q)}{2}}}$$

energy estimate

estimate (v)

gives est of $\|f(u)\|_{L^2}^2$ through $\left(|u|^{R-1}, |\nabla u|^2\right)$ in some power (Less than 1) $\Rightarrow$ put this in RHS of (IV) gives the desired estimate $\square$

Why $s<4$ is important?

Steady N-S problem:

$$\begin{cases} -\Delta u + \nabla p + f(u) + (u,\nabla)u = g \\ u|_{\partial\Omega} = 0, \ \operatorname{div} u = 0 \end{cases}$$

(1) Energy estimate for $\|f(u)\|_{L^2}$ does not change since the inertial term vanishes.

(2) Treat $(u,\nabla)u$ as a perturbation $\Rightarrow$ put it into the RHS

$$\|(u\cdot\nabla)u\|_{L^2} \le \|u\|_{L^\infty}\cdot\|\nabla u\|_{L^2} \le \|\nabla u\|_{L^2}^{3/2}\cdot\|u\|_{H^2}^{1/2}$$

(m2) reads — energy estimate

$$\|u\|_{H^2}^2 + \|\nabla p\|_{L^2}^2 + \|f(u)\|_{L^2}^2 \le Q(\|g\|_{L^2})\left(1+\|u\|_{H^2}^{3/2}\right)$$

$3/2 < 2$ allows us to complete the estimate

$$\|u\|_{H^2}^2 + \|\nabla p\|_{L^2}^2 + \|f(u)\|_{L^2}^2 \le C\left(1+\|g\|_{L^2}^2\right)^N \quad (m\,m)$$

Max-Reg for steady NSBF-equations.

dream: $N=1$ - not obtained yet.

Application to non-stationary case:

$$(n)\quad \begin{cases} \partial_t u + (u,\nabla)u + f(u) + \nabla p = \Delta u + g(t) \\ u|_{\partial\Omega} = 0\,,\ \operatorname{div} u = 0\,,\ u|_{t=0} = u_0 \end{cases}$$

$$g'(t) \in L^2_{loc}(\mathbb{R}_+, L^2)$$

① Differentiate (n) in time: $v = \partial_t u$

$$\partial_t v + (v,\nabla)u + (u,\nabla)v + f'(u)v + \nabla\tilde{p} = \Delta v + g'(t)$$

$\times\, v$ and integrate in space

$((u,\nabla)v, v) \equiv 0$ $\quad |((v,\nabla)u, v)| \le (|u|\cdot|v|, |\nabla v|) \le$

$\le \varepsilon \|\nabla v\|^2 + \varepsilon(|u|^{R-1}, |v|^2) + C\|v\|^2_{L^2}$ - under control

$$\Rightarrow \boxed{\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \|\nabla v\|^2_{L^2} + (|u|^{R-1}, v^2) \le C\left(1 + \|g'(t)\|^2_{L^2}\right)}$$

$\Rightarrow$ $\|\partial_t u(t)\|^2_{L^2}$ is under control

Finally, rewrite (n) as a steady NSBF-equation

$$-\Delta u + (u,\nabla)u + f(u) + \nabla p = g(t) - \partial_t u(t)$$

- L^2-norm is under control

t is fixed $\Rightarrow$ (nn) gives us the control of $\|u(t)\|_{H^2}$ for all t.

$\Rightarrow$ well-posedness of (n) in H^2 $\square$