Lecture 6. Stokes operator revisited.

Standard theory:

$$\begin{cases} -\Delta u + \nabla p + g = 0 \\ u|_{\partial\Omega} = 0, \ \operatorname{div} u = 0 \end{cases}$$

Spaces:

$$V = \{ u \in C_0^\infty(\Omega), \ \operatorname{div} u = 0 \}$$

$$H = [V]_{[L^2(\Omega)]^3}$$

$$V = [V]_{[H_0^1(\Omega)]^3}$$

if Ω - bounded and good enough

$$H = \{ u \in L^2(\Omega), \ \operatorname{div} u = 0, \ u \cdot n|_{\partial\Omega} = 0 \}$$

$$V = \{ u \in H_0^1(\Omega), \ \operatorname{div} u = 0 \}$$

V - is a dense subset of H:

V^* = completion of H w.r.t. $\|u\|_{V^*} := \sup\limits_{v \in V} \frac{|(u,v)|}{\|v\|_V}$

Gelfand's triple:

$$V \subset H \subset V^*$$

All embeddings are compact and dense!

Weak formulation: for a given $g \in V^*$ find $u \in V$ such that

$$[u, \varphi] = (g, \varphi) \quad \forall\, \varphi \in V$$

RRT: $\exists!$ solution of this problem $u = A^{-1} g$

where $A : V \to V^*$ is such that

$$[u, \varphi] = (Au, \varphi)$$

$[u, \varphi] := (\nabla u, \nabla \varphi)$

$(u, \varphi) = \int_\Omega u \cdot \varphi \, dx$

(u, φ) is extended to the duality between V and V^*.

A exists also by RRT – **Stokes operator**

can be considered as an unbounded operator in V^* with the domain $D_{-1}(A) = V$

A is positive definite, self-adjoint with compact inverse.

Also can be considered as an unbounded operator in H with the domain

$$D(A) = \{ u \in H^2(\Omega) \cap H_0^1(\Omega),\ \operatorname{div} u = 0 \}$$

May be not true for non-smooth Ω!

Connection with the Laplacian:

$A = -\Pi\Delta$ if we define it in H.

$\Pi : [L^2(\Omega)]^3 \to H$ is a Leray projector (ortho projector to div-free vector fields)

Well-known: Π - zero order elliptic PDO.

Fractional powers: A is self-adjoint, positive in H with compact inverse + Hilbert-Schmidt:

$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ - eigenvalues of A with complete orthonormal base $\{e_k\}_{k=1}^{\infty}$

$Ae_k = \lambda_k e_k$ $\quad \boxed{\|u\|_H^2 = \sum' (u, e_k)^2}$ – Parseval

$H_A^s := \mathcal{D}(A^{s/2}) := \{ u \in H \quad \|u\|_{H_A^s}^2 := \sum' \lambda_k^s (u, e_k)^2 < \infty \}$

for $s<0$ we use completion of H w.r.t. this

$H_A^1 = V$, $H_A^{-1} = V^*$

More general: if Ω is smooth and $\boxed{0 \le s \le 2}$ – straightforward by interpolation

$H_A^s = \mathcal{D}((-\Delta)^{s/2}) \cap H$

↑ fractional powers of the Laplacian ↖ can be identified with Sobolev spaces.

Less known facts: important for the non-steady stokes problem?

Q: What is V^*? How is it related with $H^{-1}(\Omega) \cap \{ \operatorname{div} u \} = 0$

We know that if $g \in H^{-1}(\Omega)$ then problem

$$\begin{cases} -\Delta u + \nabla p = g \\ u|_{\partial\Omega} = 0, \ \operatorname{div} u = 0 \end{cases}$$

possesses a unique solution $u \in V$, $p \in L^2(\Omega)$

Common mistake: to conclude from here that

$$\boxed{H_A^{-1} = V^* = \{ u \in H^{-1}(\Omega), \ \operatorname{div} u = 0 \}}$$ ✗

✗

Actually, this is wrong! H_A^{-1} is not a subset of distributions! Moreover, if we define $\tilde{A}$ as an unbounded operator in $H^{-1}(\Omega) \cap \{\operatorname{div} u = 0\}$ with the domain $D(\tilde{A}) = V$, then

$\tilde{A}$ is not closable $\Rightarrow$ H_A^{-1} is not a subspace of distributions!

But it is naturally def as a factor-space:

$$H_A^{-1} = H^{-1}(\Omega) / \nabla L^2(\Omega) = \{ \tilde{H}^1(\Omega), \operatorname{div} u = 0 \} / \nabla(\text{harmonic } L^2)$$

Consequences: problems with H^{-1}-regularity of non-steady Stokes problem:

$$\begin{cases} \partial_t u + \nabla p - \Delta u = g(t) \\ u|_{\partial\Omega} = 0, \ \operatorname{div} u = 0 \end{cases}$$

Known if $g \in L^2(\mathbb{R}, L^2(\Omega)) \Rightarrow$ maximal reg.

$$\partial_t u, \Delta u, \nabla p \in L^2(\mathbb{R}, L^2(\Omega))$$

But if $g \in L^2(\mathbb{R}, H^{-1}(\Omega))$ then

$u \in L^2(\mathbb{R}, H^1_0(\Omega))$ - energy estimate

But $\partial_t u \in L^2(\mathbb{R}, H^{-1}_A)$ - only $\Rightarrow$ $p \notin L^2(\mathbb{R}, L^2(\Omega))$

explicit regularity for p seems **not known**?

non-homogeneous problem

$$\partial_t u + \nabla p - \Delta u = 0, \ \operatorname{div} u = h(t), \ u|_{\partial\Omega} = 0$$

$$h \in L^2(\mathbb{R}, H^1(\Omega)) \cap \{\langle h \rangle_\Omega = 0\} \cap \partial_t h \in L^2(\mathbb{R}, H^{-1}(\Omega))$$

is not enough to get

$$u \in L^2(\mathbb{R}, H^2(\Omega)) \cap \partial_t u \in L^2(\mathbb{R}, L^2(\Omega))$$

here the sharp conditions to $\partial_t h$ to guarantee this regularity are known!

Addon 1: $\tilde{A}$ in $H^{-1}(\Omega)$ - proof

Spaces: $H^1_{0,\nabla} := \{ u \in H^1_0(\Omega),\ \exists p \in H^2(\Omega),\ u = \nabla p \}$

$H^{-1}_{div} := \{ u \in H^{-1}(\Omega),\ \operatorname{div} u = 0 \}$

$\mathcal{G}_0 := \{ u \in H^{-1}_{div},\ (u, \varphi) = 0 \quad \forall \varphi \in H^1_{0,\nabla} \}$

Lemma 1 (density): V is dense in $\mathcal{G}_0$ in $H^{-1}(\Omega)$-norm

Proof: By contradiction: if not $\exists\, v \in [H^{-1}(\Omega)]^* = H^1_0(\Omega)$ such that $(v, V) = 0$, but $(v, \mathcal{G}_0) \neq 0$

$(v, \varphi) = 0 \ \forall \varphi \in V \Rightarrow v = \nabla p \Rightarrow v \in H^1_{0,\nabla} \Rightarrow (v, \mathcal{G}_0) \equiv 0$ □

Counterexample ($\tilde{A}$ is not closable)

Take Φ which solves $\begin{cases} \Delta \Phi = 0, \ \partial_n \Phi|_{\partial\Omega} = g \in L^2(\partial\Omega) \\ g \neq 0 \end{cases}$ $\langle g \rangle = 0$

Then $(\nabla \Phi, u) = (\nabla \Phi, \nabla p) = -(\Delta \Phi, p) + \int_{\partial\Omega} g p \, d\mathcal{S} = 0$

since $p|_{\partial\Omega} = \text{const}$ ($\nabla p|_{\partial\Omega} = 0 \Rightarrow p|_{\partial\Omega} = \text{const}$)

$\Rightarrow \nabla \Phi \in \mathcal{G}_0$ if $\nabla \Phi \in H^{-1}(\Omega)$

Let $u_n \in V$ be such that $\| u_n - \nabla \Phi \|_{H^{-1}(\Omega)} \to 0$ $n \to \infty$

and let $v_n \in V$ solve

$$\begin{cases} -\Delta v_n + \nabla p_n = u_n \\ \operatorname{div} v_n = 0, \ v_n|_{\partial\Omega} = 0 \end{cases} \Rightarrow \boxed{u_n = A v_n}$$

(apply Π to the eqn)

On the other hand:

$$\begin{cases} -\Delta V_n + \nabla(P_n - \Phi) = u_n - \nabla\Phi \\ V_n|_{\partial\Omega} = 0, \ \operatorname{div} V_n = 0 \end{cases}$$

+ energy estimate

$$\|V_n\|_{H^1_0} \le C \|u_n - \nabla\Phi\|_{H^{-1}} \to 0 \quad n \to \infty$$

Result:

$V_n \to 0$ in $H^1_0(\Omega) \subset H^{-1}_{div}(\Omega)$, But $u_n = AV_n \to \nabla\Phi \neq 0$ in $H^{-1}(\Omega)$

$\Rightarrow$ $\tilde{A}$ is not closable

Moreover since $V_n \to 0$ in V and $AV_n = u_n$

$\Rightarrow$ $u_n \to 0$ in V^* but $u_n \to \nabla\Phi \neq 0$ in $H^{-1}(\Omega)$

V^* can not be realized as a subspace of $\mathcal{D}'(\Omega)$

[proof is "stolen" from J. Guermond and A. Salgado, J. Appl. Math. Comp (2011)] - although no statement on $\tilde{A}$ there

Conclusion: $H^{-1}_{div}(\Omega)$ and V^* are different spaces!

Both inclusions $H^{-1}_{div} \hookrightarrow V^*$ and $V^* \hookrightarrow H^{-1}_{div}$ are wrong!

But we have natural maps $\Pi: H^{-1}_{div} \to V^*$ and $\Pi: H^{-s} \to V^*$ as canonical inclusions to factor spaces. In particular $A = -\Pi\Delta$ in V as well: $\Delta: H^1_0 \to H^{-1}$, $\Pi: H^{-1} \to V^*$

But if $s > -1/2$, we still have $H^s_A = H^s_{div} \cap \{u.n|_{\partial\Omega} = 0\}$