Lecture 7 Navier-Stokes-BF: non-steady case

Equation:

$$\begin{cases} \partial_t u + (u,\nabla)u + \nabla p + f(u) - \Delta u = g(t) \\ u|_{\partial\Omega} = 0, \quad \operatorname{div} u = 0, \quad u|_{t=0} = u_0 \end{cases} \qquad \Omega \subset\subset \mathbb{R}^3$$

Assumptions:

$$\varkappa |u|^{R-1} - C \le f'(u) \le C(1+|u|^{R-1}), \quad \varkappa > 0, \quad R > 3$$

$$g \in L^2_{loc}([0,\infty), L^2(\Omega))$$

$$u_0 \in H$$

+ extra assumption $\qquad f(u) = \nabla_u F(u)$

Energy estimates: $\times u$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \frac{\varkappa}{2}\|u\|^{R+1}_{L^{R+1}} + \|\nabla u\|^2_{L^2} \le C\left(1 + \|g(t)\|^q_{L^q}\right)$$

$$\Rightarrow \|u(t)\|^2_H + \int_0^t e^{-\lambda(t-s)}\left(\|\nabla u(s)\|^2_{L^2} + \|u(s)\|^{R+1}_{L^{R+1}} + \|f(u(s))\|^q_{L^q}\right) ds \le \qquad (*)$$

$$C\|u_0\|^2_H e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\left(\|g(s)\|^q_{L^q} + 1\right) ds$$

just the Gronwall inequality, under control

Estimate for the inertial term:

$$\|(u,\nabla)u\|^q_{L^q} \le \|u\|^q_{L^{2q/2-q}} \cdot \|\nabla u\|^q_{L^2} \le \|\nabla u\|^2_{L^2} + \|u\|^{2q/2-q}_{L^{2q/2-q}} \le C\left(\|\nabla u\|^2_{L^2} + \|u\|^{R+1}_{L^{R+1}}\right)$$

since $\quad 2q/2-q = \frac{2(R+1)}{R-1} \ge R+1 \quad$ if $R \ge 3$.

L_q-regularity for linear Stokes ($f(u)$ and $(u,\nabla)u$ are treated as perturbations)

$\Rightarrow$ gives us the control of

$$\int_0^t e^{-\alpha(t-s)}\left(\|u(s)\|^q_{W^{2,q}} + \|\nabla p(s)\|^q_{L^q} + \|\partial_t u(s)\|^q_{L^q}\right)ds \le$$

$$C\left(\|u_0\|^2_{L^2} + \|u_0\|^q_{W^{2(1-\frac{1}{q}),q}}\right)e^{-\alpha t} + C\int_0^t e^{-\alpha(t-s)}\left(1+\|g(s)\|^q_{L^q}\right)ds \qquad (**)$$

(*) and (**) is a starting point for non-linear localization

Aim: to get the H^1-analogues of these estimates:

Step 1: interior estimates (very similar to the steady case)

$\times\ \partial_{x_i}(\varphi^2\partial_{x_i}u)$

$$\frac{d}{dt}(\varphi^2,|\nabla u|^2) + (\varphi^2,|\Delta u|^2) + \mathfrak{Re}(|u|^{R-1},|\nabla u|^2) \le C(1+\|g(t)\|^2_{L^2}) +$$

$$C\left(\|\nabla u\|^2_{L^2} + \|u\|^{R+1}_{L^{R+1}} + \|\nabla p\|^q_{L^q}\right) + 2(\Delta_x p, \nabla\varphi^2\cdot u) - 2((u,\nabla)u, \operatorname{div}(\varphi^2\nabla u))$$

+

$$\Delta p = -\operatorname{div} f(u) - \operatorname{div} g - \operatorname{div}((u,\nabla)u)$$

two extra terms coming from the inertial term

Both extra terms are easy to estimate!

$$|((u,\nabla)u, \operatorname{div}(\varrho\nabla u))| \le C(|u|\cdot|\nabla u|, \varrho|\Delta u|) + C(|u|, \varrho^{1/2}|\nabla u|^2) \le$$
$$\le \frac{1}{4}(\varrho, |\Delta u|^2) + \frac{1}{4}\varepsilon(|u|^{R-1}, |\nabla u|^2) + C\|\nabla u\|_2^2.$$

And the term $(\operatorname{div}((u,\nabla)u), \nabla\varrho\cdot u)$ is estimated analogously.

$\Rightarrow$ we get the desired interior estimate:

$$(\varrho, |\nabla u(t)|^2) + \int_0^t e^{-\alpha(t-s)}\big((\varrho, |\Delta u(s)|^2) + (|u|^{R-1}, \varrho|\nabla u(s)|^2)\big)\,ds \le$$
$$\le \|u_0\|_V^2 e^{-\alpha t} + C\Big(1 + \int_0^t e^{-\alpha(t-s)}\|g(s)\|_{L^2}^2\,ds\Big)$$

Step 2. Boundary estimates: tangential directions.
Similar to the steady case, but much more technical, mainly due to the inertial term. Only some key points will be explained, see the preprint for details.

We start with multiplication by $\partial_t^* \partial_t u$ here the inertial term is not dangerous since

$$|(\partial_t((u,\nabla)u), \partial_t u)| = |((\partial_t u, \nabla)u, \partial_t u)| \le C(|\partial_t u|\cdot|u|, |\nabla(\partial_t u)|) \le$$
$$\le \varepsilon\|\nabla(\partial_t u)\|_{L^2}^2 + C_\varepsilon(|u|^2|\nabla u|^2) \le \varepsilon(\|\nabla\partial_t u\|_{L^2}^2 + (|u|^{R-1}, |\nabla u|^2)) +$$
$$+ C_\varepsilon\|\nabla u\|_{L^2}^2, \quad \varepsilon << 1 \quad \text{since} \quad R > 3$$

As a result, we get

$$\frac{d}{dt}\|\partial_z u\|_{L^2}^2 + \alpha\left(\|\nabla\partial_z u\|^2 + (|u|^{R-1}|\nabla u|^2) + \|\partial_z f(u)\|_{L^q}^q\right) \le C\left(1+\|g(t)\|_{L^2}^2\right) +$$

$$+ C\|\nabla u\|_{L^2}^2 + C\|u\|_{L^{R+1}}^{R+1} + C\left|(A(x)\nabla(\partial_z p), u)\right|$$

exactly as in the steady case. So we only need to estimate the pressure term. Since we do not have max-regularity for the RHS $\in L^2(H^{-1})$ and due to problems with max-Reg for non-zero div, we need to be more accurate.

① We introduce $G(t)$ as a solution of linear problem

$$\partial_t G + \nabla P_G - \Delta G = g(t), \quad G|_{t=0} = u_0, \quad \operatorname{div} G = 0, \quad G|_{\partial\Omega} = 0$$

max-Reg $\Rightarrow$

$$\int_0^t e^{-\alpha(t-s)}\left(\|\partial_t G(s)\|_{L^2}^2 + \|\Delta G(s)\|_{L^2}^2 + \|\nabla P_G(s)\|_{L^2}^2\right) \le$$

$$\le \|u_0\|_H^2 e^{-\alpha t} + C\int_0^t e^{-\alpha(t-s)}\|g(s)\|_{L^2}^2. \quad \alpha > 0$$

$\Rightarrow$ G is regular enough and does not cause any problems.

Let $V = u - G$, $W = \partial_z V \Rightarrow$

$$\partial_t W - \Delta W + \nabla P_W = \partial_z f(u) - \partial_z((u,\nabla)u) - [\Delta, \partial_z]V + [\nabla, \partial_z]P_V, \quad W|_{\partial\Omega} = 0$$

$$W|_{t=0} = 0, \quad \operatorname{div} W = H(t) = [\operatorname{div}, \partial_z]V$$

Want to apply L_q-regularity est to equation on W. The terms $\partial_t f(u)$, $[\Delta, \partial_t] v$, $[\nabla, \partial_t p_v]$ do not cause any problems, but $H(t)$ and $(\partial_t u, \nabla) u$ do!!

(A) Non-homogeneous divergence: $H(t)$ can be controlled only in $L^q(0,t; H^1) \cap W^{1,q}(0,t; H^{-1})$ - not enough for the L_q-regularity! Solution:

$$H(t) = \operatorname{div} W(t) + h(t), \quad W_i = -v \cdot \nabla z_i, \quad h = v \cdot \nabla \operatorname{div} z$$

Crucial: $W|_{\partial\Omega} = 0$ and h is more regular. For such functions $H(t)$ the L_q-regularity holds.

(B) The term $(\partial_t u \nabla) u$ requires a lot of work

$$\|(\partial_t u, \nabla) u\|_{L^q}^q \leq C \|\nabla(\partial_t u)\|_{L^2}^2 + C \|\partial_t u\|_{L^2}^{5q-4/3-q} \cdot \|\partial_t u\|_{L^2}^{2(2-q)/3-q} \cdot \|u\|_{W^{2,q}}^{q/3-q}$$

this is an accurate Hölder and interpolation $\leq$

After that we integrate in time and take the L^∞-norm of the first $\partial_t u$ in time. The fact that

$$\frac{5q-4}{3-q} < 2 \quad \text{allows us to close the estimate}$$

$$q < 4/3 \qquad m := 2 - \frac{5q-4}{3-q} > 0$$

and get the estimate:

$$\int_0^t e^{-B(t-s)}\left(\|\nabla(\partial_t u(s))\|_{L^2}^2 + (|u|^{R-1}, |\partial_t u(s)|^2)\,ds + \|\partial_t f(u(s))\|_{L^q}^q\right)ds +$$

$$+ \|\nabla(\partial_t u(t))\|_{L^2}^2 \le C\left(\|u_0\|_V^2 e^{-Bt/2} + \int_0^t e^{-B(t-s)/2}\|g(s)\|_{L^2}^2 ds + 1\right)^{\widetilde{m}}$$

where $\widetilde{m} = \max\{1, 2/m\}$

Note: $\widetilde{m} > 1$ in general. We do not know why?

Step 3: Derivation of Strichartz type estimate:

$\|u\|_{L^2(0,t;L^\infty(\Omega))}$ - postponed to the next lecture

Step 4: The final estimate. x the equation by $\partial_t u$

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 + (F(u),1) + L\|u\|_{L^2}^2\right) \le C(\|g\|_{L^2}^2 + \|u\|_{L^2}^2) + C\||u|\cdot|\nabla u|\|_{L^2}^2 \le$$

$$\le C(\|g(t)\|^2 + \|u\|_{L^2}^2) + C\|u\|_{L^\infty}^2\left(\frac{1}{2}\|\nabla u\|^2 + (F(u),1) + L\|u\|_{L^2}^2\right)$$

where L is Big enought that $(F(u),1) + L\|u\|_{L^2}^2 \ge 0$.

Strichartz type estimate + Gronzwall give the final estimate.

Let $u_0 \in \underline{\Phi} := V \cap L^{R+1}$ – natural phase space for more regular solutions.

Theorem: Let $u_0 \in \underline{\Phi}$. Then the unique weak solution of NSBF equation satisfies the estimate

$$\|u(t)\|^2_{\underline{\Phi}} + \int_{\max\{0,t-1\}}^{t} \|\partial_t u(s)\|^2_{L^2}\,ds \le Q\left(\|u_0\|^2_{\underline{\Phi}} + \int_0^t e^{-\beta(t-s)}\|g(s)\|^2_{L^2}\,ds\right)$$

where Q is a monotone increasing function and $\beta > 0$ are independent of t, u_0 and g.

Remark: 1) We also have the control for the integrals (in time) of $\|\nabla(\partial_t u(s))\|^2_{L^2}$ and $(|u(s)|^{R-1}, |\partial_t u(s)|^2)$ useful for further reg and attractors

2) The gradient structure $f(u) = \nabla F(u)$ is used at the final step only – Strichartz type estimate does not require it.

3) We do not know what to do without the gradient structure – open problem.