Lecture 8. Non-steady NS-BF equation (continued)

Problem:

$$\begin{cases} \partial_t u + (u,\nabla)u + \nabla p - \Delta u + f(u) = g(t) \\ u|_{\partial\Omega} = 0,\ u|_{t=0} = u_0,\ \operatorname{div} u = 0 \end{cases} \qquad \Omega \subset\subset \mathbb{R}^3$$

$$f'(u) \sim |u|^{R-1},\ R > 3 \qquad g \in L^2_{loc}(\mathbb{R}_+, L^2(\Omega))$$

$$u_0 \in V_R L^{R+1}(\Omega)$$

Non-linear localization:

$$\int_0^t \left(\|\nabla(\partial_\tau u(s))\|^2_{L^2} + \left(|u(s)|^{R-1}, |\partial_\tau u(s)|^2 \right) \right) e^{-\beta(t-s)} ds \sim \text{under control}$$

In the steady case we used this term

Does not work now! since $f(u)$ is not under control in the $L^2(L^2)$-norm.

Instead, we use this term + L^q-regularity from the energy estimate

Interpolation:

$$W^{1,q}(0,1; L^q((0,1)^2)) \cap L^2(0,1, W^{1,2}((0,1)^2)) \subset W^{\theta,3}(0,1; W^{1-\theta,3}((0,1)^2))$$

$$\subset L^{3q}(0,1; L^{3q}((0,1)^2)), \quad \theta = 1/3$$

Applying this to ∇u:

$$\|\nabla u\|_{L^{3q}} \le C \|u\|_{W^{2,q}}^{1/3} \cdot \|\nabla(\partial_z u)\|_{L^2}^{2/3} + L.O.T$$

1st key interpolation inequality - no more localization after this point!

2nd key interpolation inequality:

$$\|u\|_{L^\infty} \le C \|u\|_{L^{R+1}}^{1/4} \cdot \|u\|_{W^{1,3q}}^{3/4}$$ - standard

Combining them:

$$\|u\|_{L^\infty}^2 \le C \|u\|_{L^{R+2}}^{1/2} \cdot \|u\|_{W^{2,q}}^{1/2} \cdot \|\nabla(\partial_z u)\|_{L^2} \le$$

$$\le C \left(\|u\|_{L^{R+1}}^{R+1} + \|u\|_{W^{2,q}}^{q} + \|\nabla(\partial_z u)\|_{L^2}^2 \right)$$

+ integration $\Rightarrow$ the desired Strichartz-type estimate

$$\int_0^t e^{-\beta(t-s)} \|u(s)\|_{L^2}^2 \, ds$$ - under control.

$\Rightarrow$ the key result on H^1-regularity is proved.

A lot of mysterious cancelations behind! unclear why! critical case?

Discussion and remarks:

① When the gradient estimate is obtained, we have more information on the solution, e.g $\nabla u \in L^\infty(L^2)$

$\Rightarrow$ $\nabla u \in L^2_{loc}(L^{3q})$, $u \in L^{8/3}_{loc}(L^\infty)$

$\Rightarrow$ allows us to treat the inertial term as a subordinated one. Reduction to simpler problem:

$$\partial_t u + \nabla p - \Delta u + f(u) = \tilde{g}(t)$$

still not clear what to do with it, e.g. how to get more regularity if $g \in L^s_{loc}(L^3)$, $s > 2$?

② For subcritical and critical cases $R < 5$ and $R = 5$ we get max-regularity: $\partial_t u \in L^2(L^2)$, $\Delta u \in L^2(L^2)$, $f(u) \in L^2(L^2)$.

complete result!

For $R > 5$, we have only $f(u) \in L^s_{loc}(L^2)$, $s = \frac{3q-2}{2-q} < 2$.

No maximal regularity! Not clear why?

(3) How to get asymtotic compactness in $\mathcal{P}=V_R L^{R+1}$? Important, e.g. for attractors.

Energy method! Need to justify H'-energy identity

$$\frac{d}{dt}E(u(t)) + \|\partial_t u\|_{L^2}^2 + ((u,\nabla)u, \partial_t u) = (g(t), \partial_t u)$$

$$E(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 + (F(u), 1)$$

No problems here (subbordinated term)

But there are problems here!

$$\frac{d}{dt}E(u(t)) = (-\Delta u + f(u), \partial_t u)$$

the sum $-\Delta u + f(u) + \nabla p \in L^2_{loc}(L^2)$, but not separately

$\Rightarrow$ difficult to justify.

Convex analysis helps! Since $E(u)$ is convex and even uniformly convex, we have

$$(-\Delta u(t+h) + f(u(t+h)), u(t+h) - u(t)) \le E(u(t+h)) - E(u(t)) \le$$
$$\le (-\Delta u(t) + f(u(t)), \partial_t u)$$

standard convex inequality

Integrating it over $t \in [s,T]$ $\Rightarrow$

$$\frac{1}{h}\int_s^T \left(H_u(t+h),\, u(t+h)-u(t)\right) dt \le \frac{1}{h}\int_T^{T+h} E(u(t))\,dt - \frac{1}{h}\int_s^{s+h} E(u(t))\,dt$$

$$\le \frac{1}{h}\int_s^T \left(H_u(t),\, u(t+h)-u(t)\right) dt \qquad H_u(t) := -\Delta u(t) + f(u(t))$$

$h \to 0$

$$E(u(T)) - E(u(s)) = \int_s^T \left(H_u(t),\, \partial_t u(t)\right) dt$$

for almost all $0 < s < T$.

Finally, "almost all" can be removed using the uniqueness of a solution and weak lower semicontinuity of convex f-ls.

$\Rightarrow$ Energy equality $\Rightarrow$ asymptotic compactness $\Rightarrow$ existens of various attractors in $\Phi := V_R \subset L^{R+1}$. - will be not considered here.

Merry Christmas

AND

Happy New Year!!!