Part II Vishik vortices and non-uniqueness for Navier-Stokes equations.

Literature:

(1) Misha Vishik. Instability and non-uniqueness for the Cauchy problem for Euler equations of an ideal incompressible fluid. ARXive: Part I and II.

(2) D. Albritton, Elia Brue an Maria Colombo Non-uniqueness of Leray solutions of the forced Navier-Stokes equations, ARXiV

(3) D. Albritton et al. Instability and non-uniqueness for the 2D Euler equations in vorticity form after M. Vishik.

We are mainly interested in Navier-Stokes equations
- Non-trivial connection Between dynamic instability and non-uniqueness.
- The analogue of Kolmogorov's flows for $\mathbb{R}^2$ (also important for attractors)

Lecture 1: Introduction

Navier-Stokes equations in the whole space:

$$(*)\quad \begin{cases} \partial_t u + (u,\nabla)u + \nabla p = \Delta u + f(t,x) \\ \operatorname{div} u = 0 \end{cases}, \quad x \in \mathbb{R}^2 \text{ or } \mathbb{R}^3 \qquad u|_{t=0} = u_0$$

Leray-Hopf solutions: weak solutions satisfying the energy inequality:

$$u \in C_w([0,T], L^2) \cap L^2(H^1) \cap \{\operatorname{div} u = 0\} +$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \le (f(t), u(t))$$

in the sense of distributions.

After the continuity of u at $t=0$ in the strong topology of L^2 is required (in order to have energy control through the initial data)

Existence: known since 1934 (through Galerkin or Leray approximations)

Uniqueness: millenium problem (for $f \equiv 0$)

- Without energy inequality - more simple. Non-uniqueness is known for $f \equiv 0$ as well. Huge number of solutions!

Energy $E(t) = \frac{1}{2}\|u(t)\|^2_{L^2}$ - can be an arbitrary function.

Clearly ill-posed! For Leray-Hopf ???

The construction which we will consider heavily based on f (unclear what to do for $f \equiv 0$).

Main idea - scaled variables

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x), \quad f_\lambda(t,x) = \lambda^3 f(\lambda^2 t, \lambda x)$$

and the similarity variables

$$\xi = \frac{x}{\sqrt{t}}, \quad \tau = \ln t$$

$$u(t,x) = \frac{1}{\sqrt{t}} V(\tau, \xi), \quad f(t,x) = \frac{1}{t^{3/2}} F(\tau, \xi)$$

u solves NS $\Leftrightarrow$ V - solves

$$(**) \begin{cases} \partial_\tau V - \frac{1}{2}(1 + \xi \cdot \nabla_\xi) V - \Delta_\xi V + (V; \nabla_\xi) V = F \\ \operatorname{div}_\xi V = 0 \end{cases}$$

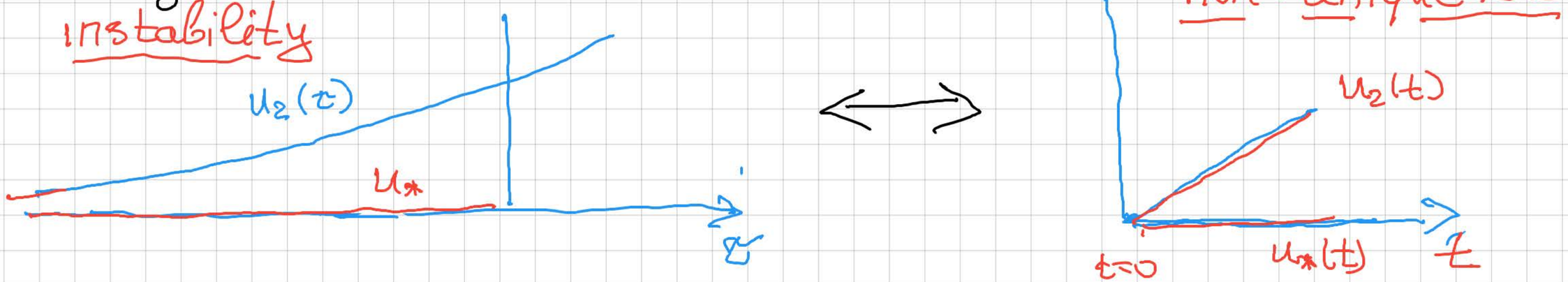
(**) - well-known transform.

Key property: The time interval $t \in (0,1)$ is transformed to $\tau \in (-\infty, 0)$

$\Rightarrow$ non-uniqueness problem for (*) $\Leftrightarrow$ finding two different solutions $U_1(\tau)$ and $U_2(\tau)$ both approaching some equilibrium $U_*(\tau)$ as $\tau \to -\infty$.

In other words: we need to find smooth (a posteriori C^∞) function F and exponentially unstable equilibrium u_* (a posteriori also C^∞, although it is not with compact support at intermediate steps)

Then: There exists an essentially unstable manifold of this equilibrium and we take $U_1 = U_*$, and U_2 is any solution on this manifold



Strategy: We need to construct exp. unstable equilibrium for the linearized problem

$$\partial_t V - \frac{1}{2}(1 + X\cdot\nabla)V - \Delta V + \nabla p = -(V,\nabla)U - (U,\nabla)V$$

where $U = U(X)$ is the equilibrium which we need to find. **Observation**: the essential spectrum of the linearized operator

$$L = \underbrace{\Delta + \frac{1}{2}(1 + X\cdot\nabla)}_{\text{symmetric, negative def}} - \underbrace{\Pi((\cdot,\nabla)U)}_{\text{relatively compact}} - \underbrace{\Pi((U,\nabla)\cdot)}_{\text{antisymmetric}}$$

Π = Leray projector

is in the left semi-plane

$$\boxed{\mathrm{Re}\,\sigma_{ess}(L) \leq 0}$$

say in $H = L^2_\sigma(\mathbb{R}^{2,3})$

$\Rightarrow$ instability can be caused by unstable eigenvalues only. $\Rightarrow$ we only need to find such an eigenvalue and $\lambda \in \sigma_p(L)$, $\mathrm{Re}\,\lambda > 0$.

$$\boxed{L e_\lambda = \lambda e_\lambda}$$

How to find the unstable profile $U(x)$

Right-Hand side $F(x)$ is in our disposal, we may just test any sufficiently smooth decaying vector field U and then define the right-hand side F via

$$F := -\Delta U + \Pi((U,\nabla)U) + \frac{1}{2}(1+x\cdot\nabla)U$$

Must be swarm of such profiles!

But, for some misterious reasons it is very difficult to find at leas one of them!

Lengthy way:

1. We start from classical 2D Euler and radially symmetric profile
2. Reduce the problem to Schroedinger operator and find the profile.
3. Pass to 3D Euler By constructing a huge vortex ring from the 2D-profile - perturbative theory
4. add $\varepsilon(\Delta U + \frac{1}{2}(1+x\cdot\nabla))$ as a singular perturbation +scale the amplitude of the ring - again perturbation theory.