Solutions

Question 1.

a) A set X with the function $d : X \times X \to \mathbb{R}$ is a metric space if the function d (metric) satisfies the following axioms:
1) $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ if and only if $x = y$.
2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3) Triangle inequality: for every 3 points $x, y, z \in X$

$$d(x, z) \leq d(x, y) + d(y, z).$$

[book]

b) By the triangle inequality

$$d(x, y) \leq d(x, u) + d(u, y) \leq d(x, u) + d(u, v) + d(v, y)$$

and, consequently,

$$d(x, y) - d(u, v) \leq d(x, u) + d(y, v)$$

Analogously

$$d(u, v) \leq d(u, x) + d(x, y) + d(y, v)$$

and

$$d(u, v) - d(x, y) \leq d(x, u) + d(y, v).$$

Thus, $|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$. [seen]

c) A subset V of a metric space X is *open* if for any point $x \in V$ there exists a ball $B_\varepsilon(x)$ of radius $\varepsilon > 0$ centered at x such that $B_\varepsilon(x) \subset V$.

A subset $V \subset X$ is *closed* if, for any sequence $x_n \in V$ which converges as $n \to \infty$ to some $x_0 \in X$, the limit point x_0 belongs to V.

Let V_α, $\alpha \in A$, be open and let

$$V = \cup_{\alpha \in A} V_\alpha.$$

Check that V is open. Let $x \in V$ be arbitrary. Then, $x \in V_{\alpha_x}$ for some $\alpha_x \in A$. Since all V_α are open, we conclude that there exists a ball $B_\varepsilon(x)$ such that

$$B_\varepsilon(x) \subset V_{\alpha_x}.$$

Thus,

$$B_\varepsilon(x) \subset V_{\alpha_x} \subset \cup_{\alpha \in A} V_\alpha = V$$

and V is open. [seen]

d) Let $f \in V$ be arbitrary and let $g \in B_\varepsilon(f)$ for some $\varepsilon > 0$. Then

$$\int_0^1 g(x)\sin x\,dx \le \int_0^1 f(x)\,dx + \int_{-1}^1 |f(x)-g(x)|\sin x\,dx \le$$
$$\le \int_{-1}^1 f(x)\sin x\,dx + 2\|f-g\|_\infty < \int_0^1 f(x)\sin x\,dx + 2\varepsilon.$$

Fix $\varepsilon = 1/2(1-\int_0^1 f(x)\sin x\,dx) > 0$. Then, from the last inequality, we see that

$$\int_0^1 g(x)\sin x\,dx < 1, \quad \text{for all} \quad g \in B_\varepsilon(f)$$

[seen] and, consequently, $B_\varepsilon(f) \subset V$ and V is open.

e) A real-valued function $x \to \|x\|$ is a norm on a vector space V if the following assumptions are satisfied:
1) $\|x\| \ge 0$ for all $x \in V$; $\|x\| = 0$ if and only if $x = 0$.
2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for all $\lambda \in \mathbb{R}(\mathbb{C})$ and all $x \in V$.
[book] 3) Triangle inequality: $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

f) To norms $\|x\|_1$ and $\|x\|_2$ on a vector space V are equivalent if there exist to positive constants l and L such that

$$l\|x\|_2 \le \|x\|_1 \le L\|x\|_2$$

for any $x \in V$.
Let $V = C[-1,1]$ and let

$$f_n(x) = \begin{cases} 1-n|x|, & |x| \le 1/n, \\ 0, & |x| > 1/n. \end{cases}$$

Then $f_n \in V$ and $\|f_n\|_{L^\infty}$, but

$$\|f_n\|_{L^1} = \int_{-1/n}^{1/n} (1-n|x|)\,dx = 2\int_0^{1/n} (1-nx)\,dx = 2(1/n - 1/(2n)) = 1/n$$

So, $\|f_n\|_{L^1} \to 0$ as $n \to \infty$. Thus and the inequality

$$\|f_n\|_{L^\infty} \le L\|f_n\|_{L^1}$$

cannot be satisfied for all n and, therefore, the space $L^1[-1,1]$ is not
[seen] equivalent to the space $C[-1,1]$.

f) Let $f \in V$ be arbitrary and let $g \in B_\varepsilon(f)$ for some $\varepsilon > 0$. Then

$$\int_0^1 g(x)\sin x\,dx \le \int_0^1 f(x)\,dx + \int_0^1 |f(x) - g(x)|\sin x\,dx \le$$
$$\le \int_0^1 f(x)\sin x\,dx + \|f-g\|_\infty < \int_0^1 f(x)\sin x\,dx.$$

Fix $\varepsilon = 1 - \int_0^1 f(x)\sin x\,dx > 0$. Then, from the last inequality, we see that

$$\int_0^1 g(x)\sin x\,dx < 1, \quad \text{for all} \quad g \in B_\varepsilon(f)$$

and, consequently, $B_\varepsilon(f) \subset V$ and V is open. [seen]

Question 2.

a) A sequence $x_n \in X$ is a Cauchy sequence in X if, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon$$

for any $n \geq N(\varepsilon)$ and any $m \in \mathbb{N}$.

A metric space (X, d) is complete if any Cauchy sequence $x_n \in X$ converges to some limit point $x_0 \in X$.

The interval $(0, 1)$ with the standard metric of $\mathbb{R}$ is a metric space which is not complete. [book]

b) A metric space (X, d) is *compact* (sequentially compact) if any sequence $\{x_n\} \in X$ contains a convergent subsequence $x_{n_k} \to x_0 \in X$ as $k \to \infty$. The space $\mathbb{R}$ with the standard metric is complete, but is not compact. [book]

c) Consider the sequence e_n of coordinate vectors in l_∞, i.e., $e_n := (\delta_{1n}, \delta_{2n}, \cdots, \delta_{nn}, \cdots)$ where δ_{ij} is a Kronecker delta. Then, for every $n \neq m$

$$\|e_n\|_{l_\infty} = \|e_m\|_{l_\infty} = 1, \quad \|e_n - e_m\|_{l_\infty} = 1.$$

Thus, $e_n \in \bar{B}_1(0)$, but the sequence $\{e_n\}_{n\in\mathbb{N}}$ cannot contain any convergent subsequence. seen

d) Let the space (X, d) be compact and let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in it. We need to prove that it is convergent to some element $x_0 \in X$. Since, X is compact, there exists a subsequence x_{n_k} of x_n convergent to some $x_0 \in X$ and we need to prove that the whole sequence x_n converges to x_0.

Since x_n is a Cauchy sequence, then for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon, \quad n \geq N, \quad m \in \mathbb{N}$$

and, in particular,

$$d(x_n, x_{n_k}) < \varepsilon, \quad \forall n \geq N \text{ and } \forall n_k > n.$$

Passing to the limit $k \to \infty$ in that inequality (which is possible to do since the metric $d(x, y)$ is a continuous function), we see that

$$d(x_n, x_0) \leq \varepsilon, \quad \forall n \geq N(\varepsilon),$$

so, $x_n \to x_0$ as $n \to \infty$. [seen]

e) The inverse function f^{-1} exists since f is one-to-one and onto. Let us prove that f^{-1} is continuous. Argue by contradiction. Assume that f^{-1} is not continuous at some point $y_0 = f(x_0) \in Y$. Then, there exists a sequence $y_n = f(x_n) \to y_0$ such that $x_n = f^{-1}(y_n)$ do not converge to $x_0 = f^{-1}(y_0)$ and, moreover,

$$d(x_n, x_0) \geq \varepsilon_0 > 0, \quad \forall n.$$

Since X is compact, there is a convergent subsequence $x_{n_k} \to x' \in X$. Passing to the limit $k \to \infty$ in the last inequality, we see that

$$d(x', x_0) = \lim_{k\to\infty} d(x_{n_k}, x_0) \geq \varepsilon_0 > 0.$$

Thus, $x' \neq x_0$. But, since f is continuous

$$f(x') = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = y_0 = f(x_0)$$

which contradicts the assumption that f is one-to-one. Thus, the continuity is proved. [unseen]

Question 3.

a) A function $f : X \to X$ on a metric space (X, d) is a contraction if there exists a number $\kappa < 1$ such that

$$d(f(x), f(y)) \leq \kappa d(x, y).$$

for all $x, y \in X$. [book]

b) The Contraction Theorem: If (X, d) is a complete metric space and f is a contraction on (X, d) then f has a unique fixed point p (i.e., the equation $f(x) = x$ has a unique solution $x = p$). [book]

c) Let $x, y \in \mathbb{R}$. Without loss of generality, we may assume that $x < y$. Then, by the finite implements formula

$$|f(x) - f(y)| = |f'(\xi)| \cdot |x - y|$$

for some point $\xi \in [x, y]$. But $0 \geq f'(\xi) = \frac{\xi^2}{1+\xi^2} < 1$ for all $\xi \in \mathbb{R}$. Thus, the inequality

$$|f(x) - f(y)| < |x - y|$$

holds.

In order to find the fixed points of $f(x)$ we need to solve the equation

$$x = f(x), \quad \text{or} \quad \frac{\pi}{2} = \arctan x$$

which does not have any real solutions. Thus, f does not have any fixed points in $\mathbb{R}$.

The function f cannot be a contraction on $\mathbb{R}$, since any contraction on the complete metric space $\mathbb{R}$ should have a (unique) fixed point (due [unseen] to the Banach theorem).

d) Indeed, let $f_1, f_2 \in C[0,1]$. Then

$$|(Tf_1)(x) - (Tf_2)(x)| = |\int_0^x y(f_1(y) - f_2(y))\,dy| \leq$$

$$\leq \int_0^1 |y||f_1(y) - f_2(y)|\,dy \leq \|f_1 - f_2\|_\infty \int_0^1 y\,dy = 1/2\|f_1 - f_2\|_\infty.$$

[seen] Thus, T is a contraction with the contraction factor $\kappa = 1/2$.

e) The fixed point $Y(x)$ which exists due to the Banach contraction theorem should satisfy

$$Y(x) = 1 - \int_0^x yY(y)\,dy, \quad x \in [0,1].$$

Differentiating this equation by x, we see that $Y'(x) = -xY(x)$ and [seen] taking $x = 0$, we see that $Y(0) = 1$.

f) Dividing the equation by Y, we have

$$\frac{dY}{Y} = -xdx, \quad \log Y = -x^2/2 + C,$$

Thus, a general solution has the form

$$Y(x) = Ce^{-x^2/2}.$$

From the initial data $Y(0) = 1$, we find $C = 1$ and, therefore, the [seen] desired fixed point is $Y(x) = e^{-x^2/2}$.

Question 4.

a) An inner product (x, y) on a vector space H is a *bi-linear*, i.e.

$$(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y), \quad a, b \in \mathbb{R},\ x_1, x_2, y \in H,$$

symmetric (i.e., $(x, y) = (y, x)$) and *positively defined* (i.e., $(x, x) \geq 0$, $(x, x) > 0$ if $x \neq 0$) form on H. A vector space V with the inner product (x, y) is a Hilbert space if it is *complete* with respect to the norm $\|x\| := (x, x)^{1/2}$. [book]

b) A system of vectors $\{e_i\}_{i=1}^{\infty}$ of a Hilbert space H is orthonormal if

$$(e_i, e_j) = \delta_{ij}$$

where the Kronecker δ_{ij} equals zero if $i \neq j$ and one if $i = j$.

An orthonormal system $\{e_n\}$ is an orthonormal basis in H if it is *complete*, i.e.

$$(x, e_i) = 0 \quad \forall j \in \mathbb{N}$$

implies that $x = 0$. [book]

c) Let $\{e_i\}$ be an orthonormal system in a Hilbert space H and let $x \in H$ be an arbitrary vector. Bessel inequality reads

$$\sum_{i=1}^{\infty} (x, e_i)^2 \leq \|x\|^2.$$

If the orthogonal system is complete (i.e., when it is a basis), the Bessel inequality is, in a fact, an equality (Parseval equality). [book]

d) Use the formulae for the Fourier coefficients

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

All a_n equal zero since the function f is odd and $\cos(nx)$ are even. So, we only need to compute b_n: integrating by parts, we have

$$\int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{1}{n} x \cos(nx)\Big|_{x=-\pi}^{x=\pi} + \frac{1}{n} \int_{-\pi}^{\pi} cos(nx)\, dx = 2\frac{\pi(-1)^{n+1}}{n}$$

and, consequently,

$$f_{lim}(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

[seen]

e) Extend the function $f(x) = x$ *periodically* from $x \in [-\pi, \pi]$ to the whole real line. Then the obtained function $f_{per}(x)$ will be piece-wise continuous and piece-wise smooth with the jump points at $x = \pi n$. By the Dirichlet theorem, the Fourier sums converge point-wise to the limit function

$$(1) \qquad f_{lim}(x) = \begin{cases} f_{per}(x), & x \neq n\pi, \\ (f_{per}(n\pi+) + f_{per}(n\pi-))/2 = 0, & x = n\pi. \end{cases}$$

This convergence cannot be uniform since the finite Fourier sums are continuous functions and the limit one is not continuous (a uniform limit of continuous functions is always continuous).

[seen]

f) Take $x = \pi/2$ in the last formula. Then $\sin(n\pi/2) = 0$ if $n = 2k$ is even and, for $n = 2k + 1$ (odd), we have $\sin((2k + 1)\pi/2) = (-1)^k$. Thus,

$$f_{lim}(\pi/2) = \pi/2 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

[seen]