Solutions

Question 1.

a) A set X with the function $d : X \times X \to \mathbb{R}$ is a metric space if the function d (metric) satisfies the following axioms:
1) $d(x, y) \geq 0$ for all $x, y \in X$; $d(x, y) = 0$ if and only if $x = y$.
2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
3) Triangle inequality: for every 3 points $x, y, z \in X$

$$d(x, z) \leq d(x, y) + d(y, z).$$

[book]

b) By the triangle inequality

$$d(x, y) \leq d(x, u) + d(u, y) \leq d(x, u) + d(u, v) + d(v, y)$$

and, consequently,

$$d(x, y) - d(u, v) \leq d(x, u) + d(y, v)$$

Analogously

$$d(u, v) \leq d(u, x) + d(x, y) + d(y, v)$$

and

$$d(u, v) - d(x, y) \leq d(x, u) + d(y, v).$$

Thus, $|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$. [seen]

c) A subset V of a metric space X is *open* if for any point $x \in V$ there exists a ball $B_\varepsilon(x)$ of radius $\varepsilon > 0$ centered at x such that $B_\varepsilon(x) \subset V$.

A subset $V \subset X$ is *closed* if, for any sequence $x_n \in V$ which converges as $n \to \infty$ to some $x_0 \in X$, the limit point x_0 belongs to V.

Let V_α, $\alpha \in A$, be open and let

$$V = \cup_{\alpha \in A} V_\alpha.$$

Check that V is open. Let $x \in V$ be arbitrary. Then, $x \in V_{\alpha_x}$ for some $\alpha_x \in A$. Since all V_α are open, we conclude that there exists a ball $B_\varepsilon(x)$ such that

$$B_\varepsilon(x) \subset V_{\alpha_x}.$$

Thus,

$$B_\varepsilon(x) \subset V_{\alpha_x} \subset \cup_{\alpha \in A} V_\alpha = V$$

and V is open. [seen]

d) A real-valued function $x \to \|x\|$ is a norm on a vector space V if the following assumptions are satisfied:

1) $\|x\| \geq 0$ for all $x \in V$; $\|x\| = 0$ if and only if $x = 0$.
2) $\|\lambda x\| = |\lambda| \cdot \|x\|$, for all $\lambda \in \mathbb{R}(\mathbb{C})$ and all $x \in V$.
[book] 3) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

e) The function $f \to \|f\|$ is *not* a norm on X since $\|f\| = 0$ for the non-zero function $f(x) \equiv 1$.

It is *a norm* on the space X_0. Indeed, since $\|f\|$ is a usual sup-norm for the derivative f', the non-negativity, 1-homogenity and the triangle inequality are satisfied. So, we only need to check that $\|f\| = 0$ implies $f = 0$.

Let f be such that $\|f\| = 0$. Then, $f'(x) \equiv 0$, $x \in [-1, 1]$. Therefore, $f(x) \equiv const$. Using that $\int_{-1}^{1} f(x)\,dx = 0$, we see that $const = 0$ and
[unseen] $f(x) \equiv 0$. Thus, $\|f\|$ is a norm on X_0.

f) The function $f_0(x) = |x|$ does not belong to V since it is not differentiable at $x = 0$. Let us prove that f_0 belongs to the closure of V in $C[-1, 1]$. Consider the sequence f_n defined by

$$f_n(x) = \begin{cases} |x|, & \text{if } \ |x| \geq 1/n \\ \frac{n}{2}x^2 + \frac{1}{2n}, & \text{if } \ |x| \leq 1/n \end{cases}$$

Then, f is continuously differentiable, $\|f_n\|_\infty \equiv 1$ and $\|f_n'\|_\infty \equiv 1$. By this reason, $f_n \in V$ for any $n \in \mathbb{N}$. On the other hand,

$$\|f_0 - f_n\|_\infty = \max_{|x| \leq 1/n} ||x|(\frac{n}{2}|x| - 1) + \frac{1}{2n}| \leq \frac{1}{n}(1/2 + 1) + 1/(2n) = 2/n$$

[seen] Thus, $f_n \to f_0$ in $C[-1, 1]$ and V is not closed.

Question 2.

a) A metric space (X,d) is *compact* (sequentially compact) if any sequence $\{x_n\} \in X$ contains a convergent subsequence $x_{n_k} \to x_0 \in X$ as $k \to \infty$. A segment $[0,1]$ with the standard metric in $\mathbb{R}$ is a compact metric space and the interval $(0,1)$ with the same metric is a metric space which is not compact. [book]

b) Hausdorff criterium: A metric space (X,d) is compact if and only if it is *complete* and *totally* bounded.

Total boundedness means that, for every $\varepsilon > 0$ the space X can be covered by the finite number of ε-balls in X. (and completness means that every Cauchy sequence in X has a limit). [book]

c) Let (X,d) be compact. Then, by Hausdorff criterium, it is totally bounded. So, for every $\varepsilon > 0$ there exists a covering of X by *finitely* many ε-balls. Fix, for every $\varepsilon > 0$ such a covering and denote by $\mathcal{C}_\varepsilon$ the union of its centers. Then, all $\mathcal{C}_\varepsilon$ are finite. Finally, set

$$V := \cup_{n\in\mathbb{N}}\mathcal{C}_{1/n}.$$

Then V is countable as a countable union of finite sets. We claim that it is dense in X. Let $x \in X$ be arbitrary. By the construction of $\mathcal{C}_\varepsilon$, there exists $x_n \in \mathcal{C}_{1/n} \subset V$ such that $d(x,x_n) < 1/n$. Thus, $x_n \in V$ and $x_n \to x$ as $n \to \infty$. Since x is arbitrary, V is dense in X and X is separable. [seen]

d) Let us prove that $(\mathbb{R},d)$ is a metric space. To this end, we need to check that $d(x,y)$ satisfies the axioms of the metric. Indeed, obviously, $d(x,y)$ equals zero iff $x = y$ and it is symmetric. So, we only need to verify the triangle inequality. Let $x,y,z \in \mathbb{R}$ be arbitrary. We need to check that

$$\min\{1,|x-y|\} \le \min\{1,|x-z|\} + \min\{1,|z-y|\}$$

If $|x-z| \ge 1$ or $|z-y| \ge 1$ then the inequality is obvious since the right-hand side is not smaller than 1 and the left-hand side is not greater than one. By, the same reason, the inequality also holds if $|x-z| < 1$ and $|y-z| < 1$, but $|x-z| + |y-z| \ge 1$. So, we only need to verify the case

$$|x-z| < 1, \ \||y-z| < 1 \ \text{ and } \ |x-z| + |y-z| < 1.$$

But, in this case the inequality reads

$$|x-y| \le |x-z| + |y+z|$$

and coincides with the triangle inequality for the standard norm in $\mathbb{R}$. Thus, (X, d) is a metric space.

This metric space is *not complete*. Indeed, consider a sequence $x_n = n$. Then,

$$d(x_n, x_m) = 1, \quad \text{if } m \neq n$$

and, therefore, this sequence cannot contain any convergent subsequences.

[seen]

e) The inverse function f^{-1} exists since f is one-to-one and onto. Let us prove that f^{-1} is continuous. Argue by contradiction. Assume that f^{-1} is not continuous at some point $y_0 = f(x_0) \in Y$. Then, there exists a sequence $y_n = f(x_n) \to y_0$ such that $x_n = f^{-1}(y_n)$ do not converge to $x_0 = f^{-1}(y_0)$ and, moreover,

$$d(x_n, x_0) \geq \varepsilon_0 > 0, \quad \forall n.$$

Since X is compact, there is a convergent subsequence $x_{n_k} \to x' \in X$. Passing to the limit $k \to \infty$ in the last inequality, we see that

$$d(x', x_0) = \lim_{k\to\infty} d(x_{n_k}, x_0) \geq \varepsilon_0 > 0.$$

Thus, $x' \neq x_0$. But, since f is continuous

$$f(x') = f(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = y_0 = f(x_0)$$

which contradicts the assumption that f is one-to-one. Thus, the continuity is proved.

[seen]

Question 3.

a) A function $f : X \to X$ on a metric space (X, d) is a contraction if there exists a number $\kappa < 1$ such that

$$d(f(x), f(y)) \leq \kappa d(x, y).$$

for all $x, y \in X$. [book]

b) The Contraction Theorem: If (X, d) is a complete metric space and f is a contraction on (X, d) then f has a unique fixed point p (i.e., the equation $f(x) = x$ has a unique solution $x = p$). [book]

c) Let $x, y \in \mathbb{R}$. Without loss of generality, we may assume that $x < y$. Then, by the finite implements formula

$$|f(x) - f(y)| = |f'(\xi)| \cdot |x - y|$$

for some point $\xi \in [x, y]$. But $0 \geq f'(\xi) = \frac{\xi^2}{1+\xi^2} < 1$ for all $\xi \in \mathbb{R}$. Thus, the inequality

$$|f(x) - f(y)| < |x - y|$$

holds.

In order to find the fixed points of $f(x)$ we need to solve the equation

$$x = f(x), \quad \text{or} \quad \frac{\pi}{2} = \arctan x$$

which does not have any real solutions. Thus, f does not have any fixed points in $\mathbb{R}$.

The function f cannot be a contraction on $\mathbb{R}$, since any contraction on the complete metric space $\mathbb{R}$ should have a (unique) fixed point (due to the Banach theorem). [unseen]

d) Indeed, let $f_1, f_2 \in C[0, 1]$. Then

$$|(Tf_1)(x) - (Tf_2)(x)| = |\int_0^x \frac{(f_1(s) - f_2(s))}{s+1}\, ds| \leq$$

$$\leq \int_0^1 |f_1(s) - f_2(s)| \frac{ds}{s+1} \leq \|f_1 - f_2\|_\infty \int_0^1 \frac{ds}{s+1} = \log 2 \|f_1 - f_2\|_\infty.$$

Thus, T is a contraction with the contraction factor $\kappa = \log 2 < 1$. [seen]

e) The fixed point $Y(x)$ which exists due to the Banach contraction theorem should satisfy

$$Y(x) = 1 + \int_0^x \frac{Y(s)}{s+1}\, ds, \quad x \in [0, 1].$$

Differentiating this equation by x, we see that $Y'(x) = (x+1)^{-1}Y(x)$ and taking $x = 0$, we see that $Y(0) = 1$. [seen]

f) Dividing the equation by Y, we have

$$\frac{dY}{Y} = (x+1)^{-1}dx, \quad \log Y = \log(x+1) + C,$$

Thus, a general solution has the form

$$Y(x) = C(x+1).$$

From the initial data $Y(0) = 1$, we find $C = 1$ and, therefore, the desired fixed point is $Y(x) = x + 1$. [seen]

Question 4.

a) An inner product (x, y) on a vector space H is a *bi-linear*, i.e.

$$(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y), \quad a, b \in \mathbb{R},\ x_1, x_2, y \in H,$$

symmetric (i.e., $(x, y) = (y, x)$) and *positively defined* (i.e., $(x, x) \geq 0$, $(x, x) > 0$ if $x \neq 0$) form on H. A vector space V with the inner product (x, y) is a Hilbert space if it is *complete* with respect to the norm $\|x\| := (x, x)^{1/2}$. [book]

b) A system of vectors $\{e_i\}_{i=1}^{\infty}$ of a Hilbert space H is orthonormal if

$$(e_i, e_j) = \delta_{ij}$$

where the Kronecker δ_{ij} equals zero if $i \neq j$ and one if $i = j$.

An orthonormal system $\{e_n\}$ is an orthonormal basis in H if it is *complete*, i.e.

$$(x, e_i) = 0 \quad \forall j \in \mathbb{N}$$

implies that $x = 0$. [book]

c) Let $\{e_i\}$ be an orthonormal system in a Hilbert space H and let $x \in H$ be an arbitrary vector. Bessel inequality reads

$$\sum_{i=1}^{\infty} (x, e_i)^2 \leq \|x\|^2.$$

If the orthogonal system is complete (i.e., when it is a basis), the Bessel inequality is, in a fact, an equality (Parseval equality). [book]

d) Use the formulae for the Fourier coefficients

$$a_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,\ \ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

All a_n equal zero since the function f is odd and $\cos(nx)$ are even. So, we only need to compute b_n: integrating by parts, we have

$$\int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{1}{n} x \cos(nx)\Big|_{x=-\pi}^{x=\pi} + \frac{1}{n} \int_{-\pi}^{\pi} cos(nx)\, dx = 2\frac{\pi(-1)^{n+1}}{n}$$

and, consequently,

$$f_{lim}(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

[seen]

e) Extend the function $f(x) = x$ *periodically* from $x \in [-\pi, \pi]$ to the whole real line. Then the obtained function $f_{per}(x)$ will be piece-wise continuous and piece-wise smooth with the jump points at $x = \pi n$. By the Dirichlet theorem, the Fourier sums converge point-wise to the limit function

$$(1) \qquad f_{lim}(x) = \begin{cases} f_{per}(x), & x \neq n\pi, \\ (f_{per}(n\pi+) + f_{per}(n\pi-))/2 = 0, & x = n\pi. \end{cases}$$

This convergence cannot be uniform since the finite Fourier sums are continuous functions and the limit one is not continuous (a uniform limit of continuous functions is always continuous).

[seen]

f) Take $x = \pi/2$ in the last formula. Then $\sin(n\pi/2) = 0$ if $n = 2k$ is even and, for $n = 2k+1$ (odd), we have $\sin((2k+1)\pi/2) = (-1)^k$. Thus,

$$f_{lim}(\pi/2) = \pi/2 = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

and

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

[seen]