MAT3004. ClassTest 1 (12% of your final mark)
Thursdayday, March 16th, 2017
Duration: 50 min. Starttime: 4pm

Problem 1.

a) (1 point). Define what does it mean that $\|\cdot\|$ is a *norm* on a vector space X.

b) (1 point). Define what is a *closure* $\bar{V}$ of a set V in in a metric space.

c) (2 points). Define what metric spaces are called *separable*. Give an example of a non-separable metric space. Justify your answer.

Problem 2. (4 points). Let $f(x) := \frac{1}{\sqrt{1-x^2}}$. Does this function belong to the space $L_1(-1,1)$? Justify your answer.

Problem 3. (4 points). Let $X := C[0,1]$ and let

$$\|f\|_1 := \int_0^1 (1-\cos x)|f(x)|\,dx, \quad \|f\|_2 := \int_0^1 x^2|f(x)|\,dx.$$

Are these norms equivalent? Justify your answer.

Solutions

Problem 1. a) A function $x \to \|x\|$ is a norm on a vector space X if the following properties hold:

i) Positivity: $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ iff $x = 0$;

ii) Homogeneity: $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$;

iii) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

b) A closure $\bar{V}$ of V in X is a union of all limit points of V in X, i.e., $x_0 \in X$ iff there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset V$ such that $x_n \to x_0$ in X.

c) A metric space (X, d) is separable if there exists a countable dense set in it. Otherwise the space is not separable. The simplest example of a non-separable metric space is a set $\mathbb{R}$ endowed by the totally disconnected metric ($d(x, y) = 1$ if $x \neq y$). Indeed, assume that V is a countable dense set in it. Then, since the only convergent sequences are stationary ones, we should have $V = \mathbb{R}$, but $\mathbb{R}$ is not countable and this contradiction shows that $(\mathbb{R}, d)$ is not separable.

Actually, there are many more interesting non-separable spaces, e.g., l_∞, $C(0, 1]$, etc. But the given example is the simplest one.

Problem 2. The function $f(x)$ is continuous for $x \in (0, 1)$ and has singularities at the end points $x = \pm 1$. Thus, due to the criterion, we should examine the following integral

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} := \lim_{\varepsilon \to 0} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{dx}{\sqrt{1 - x^2}}.$$

As in the mock-test, the anti-derivative can be found explicitly: $F(x) = \arcsin(x)$ (table integral which you are expected to know), so the integral can be found explicitly

$$I = \lim_{\varepsilon \to 0} \arcsin(x)\Big|_{x=-1+\varepsilon}^{1-\varepsilon} = \lim_{\varepsilon \to 0} 2 \arcsin(1 - \varepsilon) = \pi < \infty.$$

Thus, $f \in L^1(-1, 1)$.

Alternatively, you may not use the explicit form of the anti-derivative to answer this question. Indeed, if we separate the singularities and use that the function $f(x)$ is even, we get

$$I = \lim_{\varepsilon \to 0} \left(\int_{-1+\varepsilon}^{0} \frac{dx}{\sqrt{1 - x^2}} + \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{1 - x^2}} \right) = 2 \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{1 - x^2}},$$

so we only need to verify that the last integral is finite. To this end we use that $(1 + x) \geq 1$ for $x \in [0, 1]$ and therefore

$$\frac{1}{\sqrt{1 - x^2}} = \frac{1}{\sqrt{(1 + x)(1 - x)}} \leq \frac{1}{\sqrt{1 - x}}.$$

Thus,

$$I \leq J := 2 \lim_{\varepsilon \to 0} \frac{dx}{\sqrt{1 - x}} = 4 \lim_{\varepsilon \to 0} \sqrt{1 - x}\Big|_{x=0}^{x=1-\varepsilon} = 4 < \infty$$

and again $f \in L_1(-1, 1)$. This way looks a bit more complicated, but it is much more general since does not utilize the explicit form of the anti-derivative (which can be found only in exceptional cases!). So, if you cannot find the antiderivative in a straightforward way, just switch to this alternative approach and do not waste time on trying to find it!

Problem 3. Let us look on the quotient of weights

$$\varphi(x) := \frac{1-\cos x}{x^2}.$$

This function is continuous and positive on the interval $x \in (0,1]$ and may have singularity at $x = 0$ only. Moreover,

$$\lim_{x\to 0}\varphi(x) = \lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{x^2/2 + O(x^4)}{x^2} = 1/2.$$

Thus, the singularity at $x = 0$ is removable, so $\varphi \in C[0,1]$ and positive for all $x \in [0,1]$. Since a continuous function achieves its minimal and maximal values on the closed segment $[0,1]$, there are two positive constants $l, L > 0$ such that $l \le \varphi(x) \le L$ for all $x \in [0,1]$. Let now $f \in C[0,1]$ be an arbitrary function. Multiplying the last inequality on a positive number $x^2|f(x)|$, we get

$$lx^2|f(x)| \le (1-\cos x)|f(x)| \le Lx^2|f(x)|, \quad x \in [0,1].$$

Finally, integrating this inequality, we arrive at $l\|f\|_2 \le \|f\|_1 \le L\|f\|_2$ and since the function f is arbitrary, the norms are equivalent.