ClassTest I. (Introduction to) Function Spaces. MAT3004/MAT3010
November 24, 2011

Problem 1.

a)[2 points] Give the definition of the Lebesgue space $L^1([-1,1])$.

b)[4 points] Does the function $f(x) := \frac{1}{\sqrt{1-x^2}}$ belong to $L^1([-1,1])$? Justify your answer.

Problem 2.

a)[2 points] Give the definitions of an open set and a closed set in a metric space (X, d).

b)[2 points] Prove by first principles that the union of two open sets is open.

Problem 3. Consider the space l_2 of square summable sequences with the standard norm and let $\{e_n\}_{n=1}^{\infty}$ be the standard coordinate basis in l_2:

$$e_n = (0, 0, \cdots, 1 \text{ on } n\text{th position}, 0, \cdots).$$

a)[1 point] Let $y := (1, 1/2, 1/4, 1/8, \cdots)$. Compute $\|y\|_{l_2}$.

b)[4 points] Prove that, for every $x = (x_1, x_2, \cdots) \in l_2$,

$$x = \lim_{N\to\infty} \sum_{n=1}^{N} x_n e_n,$$

where the limit is taken in the metric of l_2.

Solutions

Problem 1

a) The Lebesgue space $L^1([a,b])$ is a completion of $C[a,b]$ (space of continuous functions) with respect to the norm

$$\|f\|_{L^1} := \int_a^b |f(x)|\,dx$$

b) According to the criterium, we need to check whether or not the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$ exists and finite as an improper Riemann integral. The function $f(x) = frac1\sqrt{1-x^2}$ is even and has two singular points $x = \pm 1$. Thus,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx =$$
$$= 2\lim_{\varepsilon\to 1}\int_0^{\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx = 2\lim_{\varepsilon\to 1}\tan^{-1}(\varepsilon) = \pi < \infty$$

Thus, $f \in L^1$.

Problem 2

a) A set $V \subset X$ is open if for every $x \in V$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset V$. A set V is closed if for every sequence $x_n \in V$ such that $x_n \to x \in X$, we have $x \in V$.

b) Let U and V be open and let $x \in U \cup V$. Then, $x \in U$ or $x \in V$ and since they are open, we have $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$ or $B_\varepsilon(x) \subset V$. In both cases, $B_\varepsilon(x) \subset U \cup V$ and $U \cup V$ is open.

Problem 3

a)

$$\|y\|_{l^2} := \left(\sum_{n=0}^{\infty} 2^{-2n}\right)^{1/2} = \left(\frac{1}{1-1/4}\right)^{1/2} = \frac{2}{\sqrt{3}}$$

b) Let $x \in l_2$ and $x^N := \sum_{n=1}^{N} x_n e_n$. We need to prove that $x^N \to x$ in l^2. Indeed,

$$\|x - x^N\|^2 = \|\sum_{n=N+1}^{\infty} x_n e_n\|^2 = \sum_{n=N+1}^{\infty} x_n^2 := S_N.$$

Since $x \in l^2$, $\sum_{n=1}^{\infty} x_n^2 < \infty$ and the series is convergent. By this reason, the tail $S_N \to 0$ as $N \to \infty$ and $x^N \to x$.