MAT3004. MockTest 1
Wednesday, March 8th, 2017
Duration: 50 min. Starttime: 11am

Problem 1.

a) (1 point). Define what does it mean that d is a *metric* on a set X.

b) (1 point). Define what is a Cauchy sequence in a metric space and what metric spaces are called *complete*.

c) (2 points). Define what metric space is called *completion* of a metric space (X, d).

Problem 2. (4 points). Let $f(x) := \frac{1}{x \ln \frac{1}{x}}$. Does this function belong to the space $L_1(0,1)$? Justify your answer.

Problem 3. (4 points). Let $X := C[0,1]$ and let

$$\|f\|_1 := \max_{x\in[0,1]} \{(1-\cos x)|f(x)|\}, \quad \|f\|_2 := \max_{x\in[0,1]} \{x|f(x)|\}.$$

Are these norms equivalent? Justify your answer.

Solutions

Problem 1. **a)** The function $d : X \times X \to \mathbb{R}$ is a metric on a set X if the following assumptions are satisfied:

i) Positivity: $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x - y$;

ii) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$;

iii) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{R}$.

b) A sequence $\{x_n\} \in X$ is Cauchy if for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $d(x_n, x_{n+m}) < \varepsilon$ for all $n > N(\varepsilon)$ and all $m \in \mathbb{N}$.

c) A metric space $(\tilde{X}, \tilde{d})$ is a completion of the metric space (X, d) if

i) $(\tilde{X}, \tilde{d})$ is complete;

ii) $X \subset \tilde{X}$ and X is dense in $\tilde{X}$ (i.e., $\bar{X} = \tilde{X}$);

iii) $d\big|_{X \times X} = d$.

Problem 2. The function $f(x)$ has two singularities: at $x = 0$ and $x = 1$ and is continuous in all other points. So, by the criterion, we need to check whether or not the following limit is finite

$$I := \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1-\varepsilon} \frac{dx}{x \ln \frac{1}{x}}.$$

There are two alternative ways of doing this. First is to compute explicitly the anti-derivative $F(x) := -\ln \ln \frac{1}{x}$, i.e., $F'(x) = \frac{1}{x \ln \frac{1}{x}}$. Then

$$I = \lim_{\varepsilon \to 0} F(1 - \varepsilon) - F(\varepsilon) = \lim_{\varepsilon \to 0} (-\ln(\ln \frac{1}{1 - \varepsilon}) + \ln \ln \frac{1}{\varepsilon}) = \infty$$

and $f \notin L_1(0, 1)$. This way is probably the simplest, but you need to find the anti-derivative explicitly which is not always possible.

Second way: you split the integral $I = I_1 + I_2$ separating the singularities

$$I_1 := \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/2} \frac{dx}{x \ln \frac{1}{x}}, \quad I_2 := \lim_{\varepsilon \to 0} \int_{1/2}^{1-\varepsilon} \frac{dx}{x \ln \frac{1}{x}}$$

and study them independently. Let us start with I_2. Using that the function $\ln x$ has a simple pole at $x = 1$, we get

$$\ln \frac{1}{x} = -\ln x = -\ln(1 - (1 - x)) = -[(1 - x) - \frac{1}{2}(1 - x)^2 - \cdots].$$

Thus, $\ln \frac{1}{x} \sim 1 - x$ near $x = 1$ and $\lim_{x \to 0} (1 - x)^{-1} \ln \frac{1}{x} = 1$. Since the function $x \to (1 - x)^{-1} \ln \frac{1}{x}$ is continuous for $[1/2, 1)$ and do not have zeros or poles on this interval, we conclude that

$$(1 - x)^{-1} x \ln \frac{1}{x} \leq \alpha > 0 \text{ and } f(x) \geq \frac{\alpha}{1 - x}$$

for $x \in [1/2, 1)$. Finally,

$$I_2 \geq \alpha \lim_{\varepsilon \to 0} \int_{1/2}^{1-\varepsilon} \frac{dx}{1 - x} = \alpha \lim_{\varepsilon \to 0} (\ln \frac{1}{\varepsilon} - \ln 2) = \infty$$

and $f \notin L_1(0,1)$. *Important:* we need not to study I_1 since the integral I will be infinite no matter finite or infinite I_1 is! So, we would need to check I_1 only if I_2 were finite. Actually I_1 is also infinite and it follows from the explicit formula for the anti-derivative.

Problem 3. This two norms are *not* equivalents. In order to see this, we look at a quotient of the weights

$$\varphi(x) := \frac{1-\cos(x)}{\sin(x)}$$

and try to find *positive* constant l and L such that $l \le \varphi(x) \le L$. If these constants are found then, obviously,

$$l\|f\|_2 \le \|f\|_1 \le L\|f\|_2$$

and the norms are equivalent. If they do not exist, the the norms are usually *not* equivalent.

In our case $\varphi(x)$ is positive and continuous for $x \in (0,1]$, so we only need to look at the neighbourhod of $x=0$. Since, $\cos(x) \sim 1 - x^2/2$, we see that $\varphi(x)$ is continuous at $x=0$ and $\varphi(0)=0$, so l being positive is impossible and we expect that norms are *not* equivalent. To prove it rigorously, we consider a sequence of functions $f_n(x) := 1-nx$ for $x \le 1/n$ and $f_n(x)=0$ otherwise and estimate the corresponding norms. Indeed, using that $1-\cos(x) \sim x^2$ near zero, we get

$$\|f_n\|_1 = \max_{x\in[0,1/n]}\{(1-\cos(x))|f_n(x)|\} \le \max_{x\in[0,1/n]}\{1-\cos(x)\} = 1-\cos(1/n) \le \alpha n^{-2}.$$

On the other hand using that the maximum is larger than the value in any point and taking the point $x_n := 1/2n$, we have

$$\|f_n\|_2 = \max_{x\in[0,1/n]}\{x(1-nx)\} \ge x_n(1-nx_n) = \frac{1}{4}n^{-1}.$$

So, the asymptotic behaviour of $\|f_n\|_1$ and $\|f_n\|_2$ is different and norms are not equivalent.