# ClassTest II. (Introduction to) Function Spaces. MAT3004/MAT3010
December 5, 2011

*Problem 1.*

**a)[2 points]** Give a definition of the contraction on a metric space $(X, d)$.

**b)[2 points]** State the Banach contraction theorem.

*Problem 2.* Let $X := C[0, \pi/4]$ be the space of continuous function with the usual sup-norm and let $F : X \to X$ be defined by

$$F(f)(x) = 1 + \int_0^x \cos(s) \cdot f(s)\, ds$$

**a)[4 points]** Prove that $F$ is a contraction on $X$.

**b)[3 points]** Find the fixed point of this map. Justify your answer.

*Problem 3.*

**a)[1 point]** State the Cauchy-Schwartz inequality for an inner product space $H$.

**b)[3 points]** Let $H := L^2([-\pi, \pi])$ be the Lebesgue space of square integrable functions with the standard inner product. Find the angle between functions $f_1(x) = x$ and $f_2(x) = \cos x$.

## Solutions

**Problem 1**

**a)** A map $F : X \to X$ is a contraction on a metric space $(X, d)$ if there exists $\kappa < 1$ such that

$$d(F(x), F(y)) \leq \kappa d(x, y)$$

for all $x, y \in X$.

**b)** Banach contraction theorem: Let $(X, d)$ be a complete metric space and let $F : X \to X$ be a contraction. Then $F$ has a unique fixed point on $X$, i.e., there exists a unique $p \in X$ such that $F(p) = p$.

**Problem 2**

**a)** Let $f, g \in C[0, \pi/4]$ be two functions. Then

$$|F(f)(x) - F(g)(x)| = |\int_0^x \cos(s)(f(s) - g(s))\, ds| \leq \int_0^x \cos(s)|f(s) - g(s)|\, ds \leq$$
$$\leq \|f - g\|_{sup} \int_0^x \cos(s)\, ds = \sin(x)\|f - g\|_{sup}.$$

Taking now supremum over $x \in [0, \pi/4$, we get

$$\|F(f) - F(g)\|_{sup} \leq \frac{1}{\sqrt{2}}\|f - g\|_{sup}$$

and $F$ is a contraction with the contraction factor $\kappa = \frac{1}{\sqrt{2}} < 1$.

**b)** The unique fixed point $y \in C[0, \pi/4]$ satisfies the integral equation

$$y(x) = 1 + \int_0^x \cos(s)y(s)\, ds$$

From this equation, we see that $y(x)$ must be continuously differentiable (since the antiderivative from the continuous function is continuously differentiable) and that $y(0) = 1$. Differentiating the equation by $x$, we end up with

$$y'(x) = \cos(x)y(x),\ \ y(0) = 1$$

and $y(x) = e^{\sin x}$.

**Problem 3**

**a)** Cauchy-Schwartz inequality: $|(x, y)| \leq \|x\|\|y\|$ for all $x, y \in V$.

**b)** To find the angle $\theta$ between $f_1$ and $f_2$ we need to use that

$$(f_1, f_2) = \|f_1\|\|f_2\| \cos\theta.$$

Then, since the function $f_1(x) = x$ is odd and the function $f_2(x) = \cos x$ is even, the product $x \cos x$ is an odd function and

$$(f_1, f_2) := \int_{-\pi}^{\pi} x \cos x\, dx = 0.$$

Thus, $\cos\theta = 0$ and $\theta = \pi/2$.