MAT3004. Test 2
Thursday, May 18th, 2017
Duration: 50 min. Starttime: 4pm

Problem 1.

a) (2 points). State the Banach contraction theorem.

b) (1 point). Define the *angle* between to non-zero vectors x and y in a Hilbert space H.

c) (1 point). State the Bessel inequality for abstract Fourier series in a Hilbert space.

Problem 2. Let $X := C[0,1]$ with the standard sup-norm and let $F : X \to X$ be defined by

$$(Ff)(x) := \frac{1}{3}f(0) + 2 + \int_0^x s^3 f(s)\,ds, \quad f \in X.$$

a) (3 points) Prove that F is a *contraction* on X.

b) (3 points) Find the *fixed point* of F.

Problem 3. (3 points). Let $H = l_2$ be the space of square summable sequences and let

$$x := \{2^{-n}\}_{n=1}^{\infty}, \quad y := \{3^{-n}\}_{n=1}^{\infty}.$$

Find the *angle* between sequences x and y in H.

Solutions

Problem 1. a) Banach Contraction Theorem: Let $F : X \to X$ be a contraction on a complete metric space (X, d). Then, there exists a unique fixed point $p \in X$ of this map.

b) The angle $\langle x, y\rangle$ between two non-zero vectors x and y of a Hilbert space H is defined by

$$\langle x, y\rangle := \arccos \frac{(x, y)}{\|x\|\|y\|},$$

where (x, y) stands for the inner product in H and $\|x\| := \sqrt{(x, x)}$.

c) Bessel inequality: Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system in a Hilbert space H. Then, for every $x \in H$, we have

$$\sum_{n=1}^{\infty}(x, e_n)^2 \leq \|x\|^2.$$

Problem 2. a) Let $f, g \in X$ be arbitrary and $x \in [0, 1]$. Then

$$|F(f)(x) - F(g)(x)| = |\frac{1}{3}(f(0) - g(0)) + \int_0^x s^3(f(s) - g(s))\, ds| \leq \frac{1}{3}|f(0) - g(0)| +$$

$$+ \int_0^x s^3|f(s) - g(s)|\, ds \leq \frac{1}{3}\|f - g\|_{sup} + \|f - g\|_{sup} \int_0^1 s^3\, ds \leq (\frac{1}{3} + \frac{1}{4})\|f - g\|_{sup}.$$

Taking the supremum over $x \in [0, 1]$ from both sides of this, we get

$$\|F(f) - F(g)\|_{sup} \leq \frac{7}{12}\|f - g\|_{sup}$$

and F is a contraction.

b) Fixed point $p \in X$ should satisfy the equation

$$p(x) = \frac{1}{3}p(0) + 2 + \int_0^x s^3 p(s)\, ds.$$

Taking $x = 0$, we see that $p(0) = \frac{1}{3}p(0) + 2$ which gets $p(0) = 3$ and differentiating the equation in x we get the equation for p:

$$p'(x) = x^3 p(x).$$

Solving this equation with this initial data, we find that $p(x) = 3e^{\frac{x^4}{4}}$.

Problem 3. Indeed,

$$\|x\|^2 = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1/4}{1 - 1/4} = \frac{1}{3}, \quad \|y\|^2 = \sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1/9}{1 - 1/9} = \frac{1}{8}$$

and analogously $(x, y) = \sum_{n=1}^{\infty} \frac{1}{6^n} = \frac{1}{5}$. Therefore $\cos(\langle x, y\rangle) = \frac{1/5}{\sqrt{1/24}} = \frac{2\sqrt{6}}{5}$ and $\langle x, y\rangle = \arccos \frac{2\sqrt{6}}{5}$.