MAT3004. MockTest 1
Wednesday, May 10th, 2017
Duration: 50 min. Starttime: 11am

Problem 1.

a) (1 point). Define what does it mean that the function $F : X \to X$ is a *contraction* on a set X.

b) (2 points). Define what is an *inner product* on a vector space and what spaces are called *Hilbert* ones.

c) (1 point). State the *Cauchy-Schwarz* inequality.

Problem 2. Let $X := C[-1/2, 1/2]$ endowed by the standard sup-norm and let $F : X \to X$ be defined by

$$(Ff)(x) := x + \int_0^x f(s)\,ds, \quad f \in X.$$

a) (3 points) Prove that F is a *contraction* on X.

b) (3 points) Find the *fixed point* of F.

Problem 3. (3 points). Let $\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$ be the Riemann zeta function. Prove that

$$\zeta(3)^2 \le \zeta(2)\zeta(4).$$

Hint: Use the Cauchy-Schwarz inequality in the space l_2.

Solutions

Problem 1. a) The function $F : X \to X$ is a contraction on a metric space (X, d) if there exists $\kappa < 1$ such that

$$d(F(x), F(y)) \le \kappa d(x, y)$$

for all $x, y \in X$.

b) A function $(\cdot, \cdot) : V^2 \to \mathbb{R}$ is called an inner product on a vector space V if the following axioms are satisfied:

i) Bi-linearity: $(\alpha x_1 + \beta x_2, y) = \alpha(x_1, y) + \beta(x_2, y)$ for all $\alpha, \beta \in \mathbb{R}$ and $x_1, x_2, y \in V$ and $(x, \alpha y_1 + \beta y_2) = \alpha(x, y_1) + \beta(x, y_2)$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y_1, y_2 \in V$;

ii) Symmetry: $(x, y) = (y, x)$ for all $x, y \in X$;

iii) Positivity: $(x, x) \ge 0$ for all $x \in V$ and $(x, x) = 0$ iff $x = 0$.

Hilbert space is an inner product space which is complete with respect to the norm $\|x\| = \sqrt{(x, x)}$.

c) Cauchy-Schwarz inequality: for any $x, y \in H$, $|(x, y)| \le \|x\|\|y\|$.

Problem 2. a) Let $f, g \in X$ be arbitrary and $x \in [-1/2, 1/2]$. Then

$$|F(f)(x) - F(g)(x)| = |\int_0^x (f(s) - g(s))\, ds|.$$

Let $x \ge 0$. Then we estimate the RHS as follows

$$|\int_0^x (f(s) - g(s))\, ds| \le \int_0^x |f(s) - g(s)|\, ds \le$$
$$\le \|f - g\|_{sup} \int_0^x ds = x\|f - g\|_{sup} \le \frac{1}{2}\|f - g\|_{sup}.$$

Let now $x < 0$. Then, analogously

$$|\int_0^x (f(s) - g(s))\, ds| \le \int_x^0 |f(s) - g(s)|\, ds \le |x|\|f - g\|_{sup} \le \frac{1}{2}\|f - g\|_{sup}.$$

Thus, in both cases, we have $|F(f)(x) - F(g)(x)| \le \frac{1}{2}\|f - g\|_{sup}$. Taking the supremum over $x \in [-1/2, 1/2]$ from both sides of this, we get

$$\|F(f) - F(g)\|_{sup} \le \frac{1}{2}\|f - g\|_{sup}$$

and F is a contraction.

b) Fixed point $p \in X$ should satisfy the equation

$$p(x) = x + \int_0^x p(s)\, ds.$$

Taking $x = 0$, we see that $p(0) = 0$ and differentiating the equation in x we get the equation for p:

$$p'(x) = p(x) + 1.$$

Solving this equation, we find that $p(x) = e^x - 1$.

Problem 3. We take two sequences $x := \{n^{-1}\}_{n=1}^{\infty}$ and $y := \{n^{-2}\}_{n=1}^{\infty}$. Both of them belong to l_2 and $\zeta(2) = \|x\|^2$, $\zeta(4) = \|y\|^2$ and $\zeta(3) = (x, y)$. By Cauchy-Schwarz inequality $(x, y) \le \|x\|\|y\|$ which gives the desired inequality.