MAT3004. Introduction to Function Spaces.
ClassTest1 (10% from your final mark)

October 31th 2014.

**Question 1 (2 points).** Define what it means for a function $d : X \times X \to \mathbb{R}$ to be the *metric* on a set $X$.

**Question 2 (1 point).** Define what it means for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $V$ to be *equivalent*.

**Question 3 (2 points).** Define what sets are called *open* and what sets are called *closed*.

**Question 4 (2 points).** Prove by first principles that the *union* of any number of open sets is open.

**Question 5 (3 points).** Let $X = C[-1, 1]$ with the standard sup-norm. Prove that the set

$$V := \{f \in X, \quad \int_{-1}^{1} x f(x)\, dx < 1\}$$

is *open* in $X$.

## Solutions

**Question 1.** A function $d : X \times X \to \mathbb{R}$ is a metric on $X$ iff it satisfies the following axioms:
1) Positivity: $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
2) Symmetry: $d(x, y) = d(y, x)$;
3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

**Question 2.** Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space $V$ are equivalent if there exist *positive* constants $l$ and $L$ such that

$$l\|x\|_1 \leq \|x\|_2 \leq L\|x\|_1$$

for all $x \in V$.

**Question 3.** A set $U \subset X$ is open if for every $x \in U$ there exists a ball $B_\varepsilon(x)$, $\varepsilon = \varepsilon(x) > 0$ in $X$ such that $B_\varepsilon(x) \subset U$. A set $V$ is closed if $V$ contains all its limit points, i.e. for any sequence $x_n \in V$ such that $x_n \to x \in X$, the limit $x \in V$.

**Question 4.** Let $U_\alpha$, $\alpha \in \mathcal{A}$ be open and let $x \in U := \cup_{\alpha \in \mathcal{A}} U_\alpha$. Then, there exists $\alpha_0 \in \mathcal{A}$ such that $x \in U_{\alpha_0}$. Since $U_{\alpha_0}$ is open, there exists a ball $B_\varepsilon(x) \in \mathcal{U}_{\alpha_0}$. Then $B_\varepsilon(x) \in U$ and $U$ is open.

**Question 5.** Consider the function $F : X \to \mathbb{R}$ defined by $F(f) := \int_{-1}^{1} x f(x)\, dx$. This function is continuous. Indeed, let $f_1, f_2 \in X$. Then

$$|F(f_1) - F(f_2)| \leq \int_{-1}^{1} |x||f_1(x) - f_2(x)|\, dx \leq \|f_1 - f_2\|_{sup} \int_{-1}^{1} |x|\, dx = \|f_1 - f_2\|_{sup}.$$

Finally, $V = F^{-1}((-\infty, 1))$ is open as an inverse image of an open set under the continuous map.