MAT3004. Introduction to Function Spaces. ClassTest II.
December 11th, 2012

Question 1 (2 points). Give the definition of the contraction on a metric space X and state the Banach contraction theorem.

Question 2. Let $f(x) = \sqrt{6+x}$ and $X = \mathbb{R}_+$ with the usual metric.

a) (2 points). Prove that f is a contraction on X.

b) (1 point). Find the limit of the following sequence:

$$\sqrt{6},\ \sqrt{6+\sqrt{6}},\ \sqrt{6+\sqrt{6+\sqrt{6}}},\ \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}},\ \cdots$$

Justify your answer.

Question 3 (1 point). State the Cauchy-Schwartz inequality.

Question 4. Let $f(x) = \operatorname{sgn}(x)$, $x \in [-\pi, \pi]$ be the sign function and

$$f(x) \sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} \tag{1}$$

be its Fourier expansions (the coefficients a_n and b_n are already computed).

a) (2 points). What is the point-wise limit of the partial sums $f_N(x)$? Is the convergence uniform? Justify your answer.

b) (2 points). Using (1), prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Hint: Use the Parseval equality.

Solutions

Question 1. A map $F : X \to X$ is a contraction on a metric space (X, d) if there exists $\kappa < 1$ such that

$$d(F(x), F(y)) \leq \kappa d(x, y)$$

for all $x, y \in X$.

Banach Contraction Theorem: Let F be a contraction on a *complete* metric space (X, d). Then, there exists a *unique* fixed point of this map $(F(p) = p)$. This fixed point can be obtained as a limit $p = \lim_{n\to\infty} x_n$ of iterations $x_{n+1} = F(x_n)$ of the map F starting from any $x_0 \in X$.

Question 2. a) According to the mean value theorem

$$|f(x) - f(y)| = |f'(\xi)| \cdot |x - y|$$

for some $\xi \in (x, y)$. Since

$$\kappa := \sup_{\xi \in \mathbb{R}_+} |f'(\xi)| = \frac{1}{2} \sup_{\xi \geq 0} \frac{1}{\sqrt{6 + \xi}} = \frac{1}{2\sqrt{6}} < 1,$$

the map is indeed contraction with the contraction factor $\kappa = \frac{1}{2\sqrt{6}}$.

b) This sequence is a sequence of iterations of the map $f(x) = \sqrt{6 + x}$ starting from $x_0 = 0$. Since the map is a contraction on the complete metric space $\mathbb{R}_+$, this sequence is convergent to the unique fixed point of that map. Thus, the limit $p >$ exists and satisfies $p = \sqrt{6 + p}$. Solving this equation, we find that $p = 3$.

Question 3. The Cauchy-Schwartz inequality: $|(x, y)| \leq \|x\|\|y\|$, where x, y are elements of a vector space V, (x, y) is an inner product on V and $\|x\|$ is the norm associated with the inner product.

Question 4. a) The function $f(x) = \operatorname{sgn}(x)$ is clearly piece-wise C^1 (even piece-wise constant) and its 2π-periodic extension has jumps at $x = \pi n$, $n \in \mathbb{Z}$. By the Dirichlet theorem, the point-wise limit of the partial sums $f_N(x)$ is equal to $f_{per}(x)$ if $x \neq \pi n$ (points of continuity) and zero for $x = \pi n$ (midvalue at jump points). The convergence is not uniform since f_{per} is not continuous.

b) The Parseval equality reads $2\pi a_0^2 + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \|f\|_{L^2}^2$ (for the classical Fourier series). In our case, $a_0 = a_n = 0$, $b_n = \frac{4}{\pi(2k+1)}$ for odd $n = 2k + 1$ and zero for even ns. The L^2-norm of f is

$$\|f\|_{L^2}^2 = \int_{-\pi}^{\pi} f^2(x)\, dx = 2\pi.$$

Thus, $\pi \sum_{k=1}^{\infty} \frac{16}{\pi^2(2k+1)^2} = 2\pi$ and $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.