MAT3004. Introduction to Function Spaces.
ClassTest2 (15% from your final mark)

December 12th 2013.

Question 1 (1 point). Define what it means for a metric space (X, d) to be *compact*.

Question 2 (1 point). State the Hausdorff criterion.

Question 3 (3 points). Let (X, d) be a compact set and let the diameter $D(X)$ be defined as follows:

$$D(X) := \sup_{x,y \in X} d(x, y).$$

Prove that there exist $x_0, y_0 \in X$ such that $D(X) = d(x_0, y_0)$.

Question 4 (3 points). Define what it means for a function $F : X \to X$ to be a *contraction* on a metric space X and *state* the Banach contraction theorem.

Question 5 (3 points). Let $X = [0, \infty)$ with the standard norm and let $f(x) := \sqrt{x + 1}$. Prove that f is a *contraction* on X.

Question 6(1 point). State the Cauchy-Schwarz inequality for vectors x and y in a Hilbert space X.

Question 7(3 points). Let $x_1, \cdots, x_n$ be real numbers. Prove that

$$\frac{x_1 + x_2 + \cdots + x_n}{\sqrt{n}} \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Solutions

Question 1. A metric space (X, d) is compact iff every sequence $x_n \in X$ has a convergent subsequence.

Question 2. Hausdorff criterion: (X, d) is compact iff it is complete and totally bounded.

Question 3. There are at least two solutions of this problem: Solution 1: Consider the function $d : X \times X \to \mathbb{R}$. This function is continuous due to the triangle inequality. Moreover, the space $X \times X$ is compact since X is compact. Therefore, the supremum of d over $X \times X$ is achieved (see the theorem in notes) at some point (x_0, y_0). Thus, $D(X) = d(x_0, y_0)$.

Solution 2: By the definition of sup, there exist sequences $x_n, y_n \in X$ such that $D(X) = \lim_{n\to\infty} d(x_n, y_n)$. Since X is compact, there exist subsequences $x_{n_k} \to x_0 \in X$ and $y_{n_k} \to y_0 \in X$. Since the metric is continuous,

$$D(X) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = d(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} y_{n_k}) = d(x_0, y_0)$$

Question 4. Let (X, d) be a metric space. A function $F : X \to X$ is a contraction on X if there exists $\kappa < 1$ such that

$$d(F(x), F(y)) \le \kappa d(x, y), \quad \forall x, y \in X.$$

Banach contraction theorem: if (X, d) is a complete metric space and $F : X \to X$ is a contraction, then F has exactly one fixed point p ($F(p) = p$) in X.

Question 5. By the mean value theorem,

$$|f(x) - f(y)| = |f'(\xi)||x - y| = \frac{1}{2\sqrt{\xi + 1}}|x - y| \le \frac{1}{2}|x - y|$$

and f is a contraction with the contraction factor $\kappa = \frac{1}{2}$.

Question 6. $|(x, y)| \le \|x\|\|y\|$, $x, y \in H$.

Question 7. We apply Cauchy Schwarz inequality to vectors $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{1} = (1, \cdots, 1)$ of $\mathbb{R}^n$ with the usual inner product. Then,

$$\sum_{k=1}^{n} x_k = (\vec{x}, \vec{1}) \le \|\vec{1}\|\|\vec{x}\| = \sqrt{n}\sqrt{\sum_{k=1}^{n} x_k^2}.$$