Some additional exercises to "(Introduction to) Function Spaces"

Part I: Metrics and norms:

1) Check whether or not the following functions are norms on the corresponding spaces:
a) $\|f\| := \sup_{x\in[0,1]} x|f(x)|$ on the space $V = C[0,1]$ of continuous functions?
b) $\|f\| := \sup_{x\in[0,1]} \frac{|f(x)-f(0)|}{x}$ on the space $C[0,1]$?
c) The previous function, but on the subspace V of $C^1[0,1]$ (continuously differentiable functions) of functions which equal zero at $x = 1/2$ ($f(1/2) = 0$)?
d) $\|f\| := \sup_{x\in[0,1]} |f(x)|$ on the space $C[-1,1]$ of continuous functions defined on a larger segment $[-1,1]$?
e) The previous norm, but on the space $H(\mathbb{C})$ of entire *analytic* functions on $\mathbb{C}$? *Hint:* Use that the function is analytic if and only if it is infinitely many times differentiable and its Taylor expantions

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

are convergent to $f(z)$ for every $z \in \mathbb{C}$.
f) $\|x\| := (\sqrt{|x_1} + \sqrt{|x_2})^2$ on $\mathbb{R}^2$.

2) Check whether or not the following norms are equivalent:
a) $\|f\|_1 := \sup_{x\in[0,1]} |f(x)|$ and $\|f\|_2 := \sup_{x\in[0,1]} x|f(x)|$ on the space of continuous functions $C[0,1]$?
b) $\|f\|_{L^p}$ and $\|f\|_{L^q}$ with $p \neq q$ on the space $C[0,1]$?
c) $\|x\|_1^2 := \sum_{n=1}^{\infty} x_n^2$ and $\|x\|_2^2 := \sum_{n=1}^{\infty} \frac{n^2}{n^2+1} x_n^2$ on the space l_2 of square summable sequences?

3) Let (X,d) be a metric space. Prove that $d_1(x,y) := \min\{1, d(x,y)\}$ and $d_2(x,y) := \frac{d(x,y)}{1+d(x,y)}$ are also metrics on X. Check also that $x_n \to x$ in metric d if and only if it is convergent in metric d_1 or metric d_2 (so these metrics are "equivalent").

4) Verify that any metric d satisfies the following inequality:

$$|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v).$$

Deduce that the metric is a continuous function from X^2 to $\mathbb{R}$ (where the set X^2 is endowed by the metric $D((x,y),(u,v)) := d(x,u) + d(u,v)$.

Part 2. Convergence, continuity and open/closed sets:

1) Let (X,d) be the space of totally disjoint points ($d(x,y) = 1$ if $x \neq y$). Describe all convergent sequences in X. What functions $f : X \to \mathbb{R}$ will be continuous? What are the open/closed sets here?

2) Prove that, in any normed space $S_\varepsilon(x_0) = \partial B_\varepsilon(x_0)$ (the boundary of an ε-ball is an ε-sphere). Give an example in a metric where it is not true.

3) Let X be a normed space and V be a subset of it such that $V \neq \varnothing$ and $V \neq X$. Prove that the boundary of V is not empty: $\partial V \neq \varnothing$. Hint: take to points $y \in V$ and $x \notin V$, consider a segment $x_s := sx + (1-s)y$ and look at x_{s^*} where $s^* := \sup\{s \in [0,1], x_s \in V\}$.

4) Prove that $x_0 \in \partial V$ iff there are two sequences $x_n \in V$ and $y_n \notin V$ such that $x_n \to x_0$ and $y_n \to x_0$. Check the following inclusions/identities:
a) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$;
b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$;
c) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$;
d) $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$;
e) $\partial(A \cup B) \subset \partial A \cup \partial B$.
In all cases, where there are inclusions construct the examples which confirm that these inclusions may be strict.
f) Check that both inclusions $\partial(A \cap B) \subset \partial A \cap \partial B$ and $\partial(A \cap B) \supset \partial A \cap \partial B$ are *not true* in general.

5) Let $f_0 \in L^2([0,1])$ and let

$$V := \{f \in L^2([0,1]), \ \int_0^1 f_0(x) f(x)\, dx < 1\}.$$

Prove in TWO different ways (by first principles and using the continuity criterium) that this set is open. You may use Cauchy-Schwartz inequality here.

6) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V and let B_1 and B_2 be open unit balls in norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Assume that the ball B_1 has an *interior* point in the space $(V, \|\cdot\|_2)$. Prove that $\|x\|_1 \leq C\|x\|_2$ for every $x \in V$ and some $C > 0$. Illustrate this property on the example of the sup and L^1-metrics on $C[0,1]$

7) Let $V = l_2$ (space of square summable sequences) and let

$$X := \{x \in V, \ \sum_{n=1}^{\infty} n^2 x_n^2 \leq 1\}.$$

Prove that X is closed. What is $\operatorname{int} X$?

Part 3. Completness and compactness:

1) Let V be a space of all FINITE sequences (the length of a sequence is not fixed) realized as a subset of l_∞ with the sup-norm (any finite sequences is completed to the infinite one by adding zeros). Explain, why V is not complete? Prove that the completion of this space coincides with the space $c_0 \subset l_\infty$ of all sequences x_n which tend to zero as $n \to \infty$. (Do not forget to prove that c_0 is closed in l_∞).

2) Let $V = C^1[0,1]$ be the space of continuously differentiable functions with the norm $\|f\|_{sup} := \sup_{x\in[0,1]} |f(x)|$. Is that space complete? Justify your answer.

3) Let $H(\bar{D})$ be the space of all functions f which are *analytic* in a unit ball $D := \{z \in \mathbb{C}, |z| < 1\}$ and *continuous* in the closed ball $\bar{D}$ with the norm $\|f\|_H := \sup_{|z|\le 1} |f(z)|$. Is that space complete? Justify your answer. Hint: you may use that the function $f \in C(\bar{D})$ is analytic iff

$$\oint_\Gamma f(z)\, d\Gamma = 0$$

for any smooth closed curve $\Gamma \subset D$.

4) Let V be a normed space, $\dim V = \infty$ and let $K \subset V$ be compact. Prove that $\operatorname{int} K = \varnothing$. You may use without proving that the closed unit ball in V is not compact.

5) Let X be compact and $f : X \to Y$ be continuous and one-to-one (X and Y are metric spaces). Prove that the inverse function $f^{-1} : Y \to X$ is also continuous.

Consider the function $f : [0, 2\pi) \to S_1 = \{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 = 1\}$ given by $f(x) = (\cos x, \sin x)$. Check that this function is one-to-one and continuous, but its inverse is discontinuous at $(1, 0)$.

6) Let $K := \{x \in l_\infty, \sup_{n\in\mathbb{N}} n|x_n| \le 1\}$. Prove in TWO different ways (using the Hausdorff Criterium and directly using the Cantor diagonal procedure) that this set is compact in l_∞.

7) Let K be a compact set in a metric space X and let

$$\operatorname{diam}(K) := \sup_{x,y\in K} d(x, y).$$

Prove that there are two points $x_0, y_0 \in K$ such that $\operatorname{diam} K = d(x_0, y_0)$.

8) Prove that any compact metric space is separable (use the Hausdorff criterium).

Part 4. Uniform, Lipschitz and Hölder continuity and Arzela theorem:

1) Let $V = C[0,1]$ and $f_n(x) = x^n$. Find the point-wise limit of that sequence. Is the convergence uniform? Justify your answer.

2) Give an example of a sequence $f_n \in C[0,1]$ such that $\|f_n\|_{sup} \le 1$ and $\|f_n - f_m\|_{sup} \ge 1$ if $n \ne m$. Can such a sequence be equicontinuous (i.e., to have the the common continuity modulus)?

3) Find (the best) modulus of continuity for the following functions:
a) $f(x) = |x|$ on $\mathbb{R}$;
b) $f(x) = x + \sin x$ on $[0, 2\pi]$;
c) $f(x) = \sqrt{x}$ on $\mathbb{R}_+$;
d) $f(x) = x^p$, $p < 1$ on $\mathbb{R}_+$;
e) $f(x) = \operatorname{sgn} x$.
Which of that functions are Lipshitz (Hölder) continuous?

3) Let $f : X \to Y$ where X is normed and Y is metric spaces be such that its modulus of continuity $\omega(z)$ satisfies $\lim_{z\to 0} \frac{\omega(z)}{z} = 0$. Prove that the function must be a constant. Hint: use that, for any $n \in \mathbb{N}$

$$d(f(x), f(0)) \le \sum_{k=0}^{n-1} d(f(\frac{k+1}{n}x), f(\frac{k}{n}x)).$$

4) Give an example of a sequence of *continuous* functions (say, on $[0,1]$) which converges to a *continuous* function point-wise, but not uniformly.

5) Let $f(x) = \cos x$, if $x \in [-\pi/2, \pi/2]$ and zero otherwise and let $f_n(x) := f(x - \pi n)$. Consider this sequence as a sequence in $C(\mathbb{R})$ of bounded and continuous functions on $\mathbb{R}$ with the standard sup-metric. Prove that this sequence is equicontinuous and uniformly bounded. Check also that $\|f_n - f_m\|_{sup} = 2$ if $n \ne m$ (and therefore it cannot contain any convergent subsequences). Does that contradict the Arzela theorem? Justify your answer.

6) Let

$$K := \{f \in C[-1,1], \quad \int_{-1}^{1} f(x)\,dx = 0, \ |f(x) - f(y)| \le |x-y|^{1/2}, \forall x, y \in [-1,1]\}$$

be a set of Hölder continuous functions with Hölder exponent $1/2$ with zero mean ($\int_{-1}^{1} f(x)\,dx = 0$). Prove that K is compact in $C[-1,1]$.