Some additional exercises to
"(Introduction to) Function Spaces: file II"

Part 5. Banach contraction theorem and applications:

1) Check whether or not the following maps are contractions on the corresponding metric spaces (if yes, find the associated fixed point):

a) $X = \mathbb{R}_+$, $f(x) = \sqrt{x+1}$?

b) $X = C[0,1]$, $F(f) := 1 + \int_0^1 sf(s)\,ds$?

c) $X = \mathbb{R}$, $f(x) = \sqrt{x^2+1}$?

d) $X := C(\mathbb{R}_+)$ (with the standard sup-norm) and $F(f)(x) := \sqrt{1+x+f(x)}$?

e) $X = \mathbb{R}$, $f(x) = \pi/2 + x - \arctan x$?

2) Let A and B be two $n \times n$ matrices such that A is *invertible* and

$$\|A^{-1}\| \cdot \|B\| < 1.$$

Recall that

$$\|A\| := \sup_{x \in \mathbb{R}^n} \frac{|Ax|}{|x|}$$

where $|\cdot|$ is a usual Euclidean norm in $\mathbb{R}^n$ (i.e., the norm $\|A\|$ of the matrix A is the least number L such that $|Ax| \le L|x|$ for all $x \in \mathbb{R}^n$). Prove that the matrix $A+B$ is also *invertible*. **Hint:** Check that the map $Fx = A^{-1}Bx + h$ is a *contraction* on $\mathbb{R}^n$ for all $h \in \mathbb{R}^n$.

3) Using the Banach contraction theorem, prove that the following sequences are convergent and find their limits:

a) $\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$.

b) $\sqrt{3},\ \sqrt{3+\sqrt{3}},\ \sqrt{3+\sqrt{3+\sqrt{3}}}, \cdots$.

Hint: Write these sequences as the iterations of some functions and check that these functions are contractions.

4) Let the map $x \to Ax$ (from $\mathbb{R}^2$ to $\mathbb{R}^2$) be given by the matrix $A := \begin{pmatrix} 1/2 & 5 \\ 0 & 1/2 \end{pmatrix}$.

a) Prove that this map is a contraction on $\mathbb{R}^2$ with the following norm:

$$\|x\| := |x_1| + 20|x_2|.$$

b) Is it a contraction on $\mathbb{R}^2$ with the standard Euclidean norm? Justify your answer.

5) The following differential equation

$$\frac{d}{dt}y(t) = 3[y(t)]^{2/3}$$

with the initial data $y(0) = 0$ has at least two solutions $y_1(t) \equiv 0$ and $y_2(t) = t^3$.

a) Does it contradict the local existence and uniqueness theorem for ordinary differential equations? Explain your answer.

b) The function $y_3(t) := (t+1)^3$ solves this equation with the initial data $y(0) = 1$ on a *half-line* $t \geq 0$. Is that solution unique? Justify your answer.

c) The function $y_3(t) := (t+1)^3$ solves this equation with the initial data $y(0) = 1$ also on the *whole line* $t \in \mathbb{R}$. Is that solution unique? Explain your answer.

6)* Let X be a *compact* metric space and $F : X \to X$ be such that

$$d(F(x), F(y)) < d(x, y), \quad x, y \in X, \ x \neq y. \tag{1}$$

a Prove that $\operatorname{diam}(F(X)) < \operatorname{diam}(X)$ (use the result of Part 3 problem 7).

b) Verify that the sequence $D_n := \operatorname{diam}(F^n(X))$ tends to zero as $n \to \infty$.

c) Check that the sequence of iterations $x_{n+1} = F(x_n)$ is a Cauchy sequence for any $x_0 \in X$ and prove that the Banach contraction theorem remains true in that case.

d) Give an example of *non-compact* complete metric space X and a function F satisfying (1) which does not have any fixed points.

7) Prove the following generalization of the Banach contraction theorem: Let (X, d) be a complete metric space and let $T : X \to X$ be a map such that the nth iteration T^n of that map is a contraction for some $n \in \mathbb{N}$. Then the map T has a *unique* fixed point in X.
Hint: Use the standard Banach contraction theorem (its proof is not required).

a) Consider the integral operator

$$(Ty)(x) := 1 + \int_0^x y(s)\, ds$$

on the Banach space $X = C[0, L]$ of continuous functions on the segment $[0, L]$ with the usual sup-norm ($L > 0$ is some fixed positive number). Prove that T is a contraction on X if $L < 1$.

b) Prove that T is not a contraction on X if $L \geq 1$.

c) Prove that the nth iteration of the map T satisfies

$$(T^n y)(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} y(s)\, ds.$$

Hint: use induction.

d) Using the previous formula, prove that, for every $L > 0$, there exists $n = n(L)$ such that the nth iteration T^n of the map T is a contraction on $X = C[0, L]$.

Part 6. Inner products and Hilbert spaces:

1) Verify what of the following functions define the inner products on the corresponding spaces?

a) $H = \mathbb{R}^2$, $(x, y) := 3x_1y_1 + 5x_2y_2$.

b) $H = \mathbb{R}^2$, $(x, y) := x_1^2 + y_2^2$.

c) $H = \mathbb{R}^3$, $(x, y) := x_1y_1 - x_2y_2 + x_3y_3$.

d) $H = C[a, b]$, $(f, g) := \int_a^b (x - a)^2 f(x)g(x)\, dx$.

e) $H = l_2$, $(x, y) := \sum_{n=1}^{\infty} n^2 x_n y_n$.

2) Let the sequences $x = \{x_n\}$ and $y = \{y_n\}$ belong to l_4. Prove that the sequence $xy := \{x_n y_n\}$ belongs to l_2 (use the Cauchy-Schwartz inequality).

3) Let $H = L^2(-1, 1)$ with the standard inner product. Find the *angles* between the following vectors:

a) $\cos \pi x$ and $\sin \pi x$.

b) x and x^2.

c) $|x|^{-1/3}$ and $|x|^{1/3}$.

4) Let H be an inner product space. Prove that the inner product (x, y) is *continuous* as a function from H^2 to $\mathbb{R}$.

5) Let H be a Hilbert space. Prove that the *parallelogram* law:

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds for any $x, y \in H$.

6)** Let H be a B-space such that the parallelogram law holds for any $x, y \in H$. Prove that $(x, y) := \frac{1}{4}(\|x + y\|^2 + \|x - y\|^2)$ is an inner product on H (so, H is a Hilbert space).

7)* *(example of a non-separable Hilbert space).* Let H be the space of all trigonometric polynomials on $\mathbb{R}$ of the form $P(x) = \sum_{n=1}^{N} a_n \cos(\lambda_n x) + b \sin(\lambda_n x)$ for $\lambda_n \in \mathbb{R}$ and let

$$(f, g) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)g(x), \quad f, g \in H. \tag{2}$$

a) Prove that (2) is well defined on H and defines an *inner product* in H.

b) Prove that the *inner product* space H *is not* separable. *Hint:* Consider the functions $e_\lambda := \sin(\lambda x)$ for all $\lambda \in \mathbb{R}$.

c) Prove that $(f, g) = \int_{-\pi}^{\pi} f(x)g(x)\, dx$ for any 2π-periodic polynomials f and g.

d) Deduce from this identity that H *is not* complete and that the completion $\tilde{H}$ of it contains the space L^2_{per} (square integrable 2π-periodic functions) as a closed subspace.

Part 7. Fourier series:

1) Let H be a Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ is an orthonormal system in it.

a) Let $H_0 \in H$ be the closure of all finite linear combinations of vectors e_n (=the linear space spanned by $\{e_n\}$). Prove that $\{e_n\}$ is a *basis* in H_0.

b) Let $x \in H$ and let $x_0 := \sum_{n=1}^{\infty}(x, e_n)e_n$ (the Fourier series of x). Prove that $x_0 \in H_0$ and that x_0 is the *orthogonal* projection of x to H_0.

2) Prove that any Hilbert space with the orthonormal basis is separable.

3) Orthogonalize the following finite sequences using the Gram-Schmidt orthogonalization:

a) $\{1, x, x^2, x^3\}$ in $L^2(-1, 1)$ (standard inner product: first Legendre polynomials).

b) $\{1, x, x^2\}$ on $\mathbb{R}_+$ with respect to $(f, g) := \int_0^{\infty} e^{-x} f(x)g(x)\, dx$ (Laguerre polynomials).

4) Let $f \in L^2([-\pi, \pi])$ and $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$.

a) Let $f_{even}(x) := \frac{f(x)+f(-x)}{2}$, $f_{odd}(x) = \frac{f(x)-f(-x)}{2}$. Prove that

$$f_{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad f_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

b) Assume that f is C^k-smooth and 2π-periodic. Prove that

$$|a_0|^2 + \sum_{n=1}^{\infty} n^{2k}(|a_n|^2 + |b_n|^2) < \infty.$$

5) a) Find the Forier expansions on $[-\pi, \pi]$ of the following functions: $f(x) = x$, $f(x) = |x|$, $f(x) = e^x$.

b) Let $f_N(x)$ be the N-th partial sums of these Fourier series. For each of that 3 functions, find the *point-wise* limit of $f_N(x)$ as $N \to \infty$ (for all $x \in \mathbb{R}$). What about the *uniform* convergence at $[-\pi, \pi]$. Justify your answer.

c) Using the above expansions, find the the expansions for $\cosh(x)$, $\sinh(x)$ and $x_+ := \max\{0, x\}$.

d) Using the above expansions, find the explicit expressions for the sums:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$

e) Write out the Bessel inequality for each that expansions. Using them, compute

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2}.$$