Hints for solutions of additional exercises.

Problem 1

a) It is a norm. All axioms can be checked straightforwardly.

b) It is not a norm by two reasons: it is not well-defined on $C[0,1]$ (look at $f(x) = \sqrt{x}$) and vanishes on constants.

c) It is a norm on $C^1[0,1]$. Unlike the previous case, it is well-defined (due to, say, finite increments formula) and $\|f\| = 0$ implies $f = 0$ (due to the condition $f(1/2) = 0$ the constant functions are excluded). All other properties of the norm are evident.

d) It is not a norm since there are non-zero continuous functions on $[-1,1]$ which equal zero for $x \in [0,1]$.

e) It is a norm, since unlike the previous example, any analytic function which equals zero for $x \in [0,1]$ must be equal zero identically (indeed, all derivatives $f^{(n)}(0)$ can be computed using only $x \in [0,1]$ and, by this reason, should be zero; from the Taylor expansions, we see that this implies $f \equiv 0$).

f) It is not a norm, since it does not satisfy the triangle inequality (for instance, you may take $x = (4,0)$ and $y = (0,4)$).

Problem 2.

a) The norms are not equivalent. Indeed, let us consider a sequence of functions f_n such that $\sup_{x\in[0,1]} |f(x)| = 1$ and $f_n(x) = 0$ if $x > 1/n$ (obviously, such a sequence exists). Then $\|f_n\|_1 = 1$ for all n, but $\|f_n\|_2 \le 1/n \to 0$ as $n \to \infty$.

b) The answer is again negative. Indeed, consider the functions $f_n(x) = 1 - nx$ for $x < 1/n$ and zero for $x > 1/n$. Then

$$\|f_n\|_{L^p} = (\int_0^{1/n} (1-nx)^p\, dx)^{1/p} = (\frac{1}{n}\int_0^1 (1-y)^p\, dy)^{1/p} = Cn^{-1/p},\ C = (p+1)^{-1/p}.$$

Thus, $\|f_n\|_{L^p} \sim n^{-1/p}$ and $\|f_n\|_{L^q} \sim n^{-1/q}$ have *different* decay rate as $n \to \infty$ which is impossible if these two norms would be equivalent.

c) These norms are equivalent. Indeed, $1/2 \le \frac{n^2}{n^2+1} < 1$ for all n and, therefore, $1/2\|x\|_1 \le \|x\|_2 \le \|x\|_1$.

Problem 3. The fact that d_1 is a metric is stated in Coursework 1 (2009), the analogous fact for d_2 can be found in Coursework 1 (2008) (for the case of $\mathbb{R}$, but the general case is the same). The equivalence follows from the fact that

$$d_2(x,y) \le d_1(x,y) \le 1/2 d_2(x,y)$$

if $d_1(x,y)$ is small enough.

Problem 4 The inequality is one of the Exam questions for year 2008 and the continuity is an immediate corollary of the inequality.

Part 2.

Problem 1. The sequence $x_n \to x_0$ in the space of disjoint points iff $x_n \equiv x_0$ for all n large enough. Indeed, if $x_n \to x_0$ than $d(x_n, x_0) \le 1/2$ for n large enough and this is possible only if $x_n \equiv x_0$. By the continuity criterium via convergent sequences, we see that ALL functions $f : X \to \mathbb{R}$ are continuous. Finally, by the definition of closed sets ANY set is closed in X and, since the complement of a closed set is open, ALL sets are also open.

Problem 2. Let $y_0 \partial B_\varepsilon(x_0)$. Then, by definition, $y_0 \in \overline{B_\varepsilon(x_0)}$ and $y_0 \notin B_\varepsilon(x_0)$ ($B_\varepsilon(x_0)$ is always open). Thus, there is a sequence $x_n \in B_\varepsilon(x_0)$ such that $x_n \to x_0$. Since, $d(x_n, x_0) < \varepsilon$ and d is continuous, we have $d(y_0, x_0) \le \varepsilon$. And we must have $d(y_0, x_0) = \varepsilon$ because $y_0 \notin B_\varepsilon(x_0)$. Thus, $y_0 \in S_\varepsilon(x_0)$ and

$$\partial B_\varepsilon(x_0) \subset S_\varepsilon(x_0)$$

for any METRIC space.

Let now $y_0 \in S_\varepsilon(x_0)$ and our space X is normed. Let $x_n := x_0 + \frac{n-1}{n}(y_0 - x_0)$. Then, $x_n \in B_\varepsilon(x_0)$ since $\|x_n - x_0| = \frac{n-1}{n}\|y_0 - x_0\| = \frac{n-1}{n}\varepsilon < \varepsilon$ and $x_n \to y_0$ as $n \to \infty$. This shows that $y_0 \in \overline{B_\varepsilon(x_0)}$. Clearly, $y_0 \notin B_\varepsilon(x_0)$ and therefore $y_0 \in \partial B_\varepsilon(x_0)$. Thus, for normed spaces

$$\partial B_\varepsilon(x_0) = S_\varepsilon(x_0).$$

Consider now the space of totally disjoint points from Problem 1. Then, $B_1(x_0) = \{x_0\} = \overline{B_1(x_0)}$, so $\partial B_1(x_0) = \varnothing$ but $S_1(x_0) = X - \{x_0\}$.

Problem 3 Let us check that the point $x_{s^*} \in \partial V$. Indeed, by the construction of s, there is a sequence of $s_n \to s^*$ such that $x_{s_n} \in V$. Then, since $x_{s_n} \to x_{s^*}$, we see that $x_{s^*} \in \bar{V}$. Assume now that $x_{s^*} \in \operatorname{int} V$. Then, there is a ball $B_\varepsilon(x_{s^*}) \in V$. In other words, all points which are *close* enough to x_{s^*} should belong to V. In particular, it should be true for $x_{s^*+\delta} = x_{s^*} + \delta(x - y)$ for all small δ. But this contradicts the choice of s^*. Thus, $x_{s^*} \notin \operatorname{int} V$ and, therefore, $x_{s^*} \in \partial V$.

Problem 4 The proofs of all inclusions are similar, so, I will do only, say, **e)**. Indeed, let $x_0 \in \partial(A \cup B)$. Thus, $x_0 \in \overline{A \cup B}$ and $x_0 \notin \operatorname{int}(A \cup B)$. According to **b)**, we see that $x_0 \in \bar{A} \cup \bar{B}$ and according to **d)**, we have $x_0 \notin \operatorname{int} A$ and $x_0 \notin \operatorname{int} B$. Thus, $x_0 \in \partial A$ or $x_0 \in \partial B$ and the inclusion is proved.

The counterexamples may be more difficult, so let us consider all of them:

a) Say, $X = \mathbb{R}$, $A = (0, 1)$, $B = (1, 2)$; **d)** Say, $X = \mathbb{R}$, $A = [0, 1]$, $B = [1, 2]$; **e)** Say, $X = \mathbb{R}$, $A = [0, 1]$, $B = [1, 2]$; **f)** Example of **a)** shows that $\partial A \cap \partial B$ may be *larger* than $\partial(A \cap B)$ and the example of two two intersecting balls in $X = \mathbb{R}^2$ shows that $\partial A \cap \partial B$ may be *smaller* than $\partial(A \cap B)$.

Problem 5.

Proof I(from first principles): Let $g \in V$. Then, $\int_0^1 f_0(x)g(x)\,dx := \delta < 1$. We need to find a ball $B_\varepsilon(g) \subset V$. Let $h \in B_\varepsilon(g)$ be arbitrary. Then

$$\int_0^1 f_0(x)h(x)\,dx = (f_0, h) = (f_0, g) + (f_0, h - g) \le \delta + \|f_0\|\|h - g\| \le \delta + \|f_0\|\varepsilon$$

(here we have used Cauchy-Schwartz). Thus, $(f_0, h) < 1$ if $\varepsilon < \frac{1-\delta}{\|f_0\|}$. Thus, V is open.

Proof II (using the continuity criterium): Consider the function $F : H \to \mathbb{R}$ where $H := L^2(0,1)$ and $F(f) := (f_0, f)$. This function is *continuous* (prove using the Cauchy-Schwartz!) and $V = F^{-1}(-\infty, 1)$. Since $(-\infty, 1)$ is open in $\mathbb{R}$ and F is continuous, V is also open.

Problem 6. Let x_0 be the interior point of B_1 in the norm of V_2. Then, by definition, $x_0 + \varepsilon B_2 \in B_1$ for sufficiently small positive ε. Thus, $B_2 \subset -\varepsilon^{-1}x_0 + \varepsilon^{-1}B_1$ and, therefore,

$$\sup_{x \in B_2} \|x\|_1 = \sup_{\|x\|_2 < 1} \|x\|_1 = \sup_{x \in V} \frac{\|x\|_1}{\|x\|_2} \le \varepsilon^{-1}\|x_0\|_1 + \varepsilon^{-1} := C$$

and, finally, $\|x\|_1 \le C\|x\|_2$.

Let now $V := C[0,1]$, $\|f\|_2 := \sup_{x \in [0,1]} |f(x)|$ and $\|f\|_1 := \int |f(x)|\,dx$. We know that $\|f\|_1 \le \|f\|_2$ and may see that the ball

$$B_1 := \{f \in V,\ \int_0^1 |f(x)|\,dx < 1\}$$

is an *open* set in the space V with the sup-norm.

Problem 7. Let $x^k := \{x_n^k\}_{n=1}^\infty \in X$ and $x^k \to x^0$ as $k \to \infty$. Then, for every fixed N, we have

$$\sum_{n=1}^N n^2 (x_n^k)^2 \le \sum_{n=1}^\infty n^2 (x_n^k)^2 \le 1.$$

Since convergence in l^2 implies the coordinate-wise convergence, then passing to the limit $k \to \infty$ in that inequality, we see that

$$\sum_{n=1}^N n^2 (x_n^0)^2 \le 1.$$

Finally, passing to the limit $N \to \infty$, we see that $x_0 \in X$ and X is closed.

Let us prove that $\operatorname{int} X = \varnothing$. Indeed, let $x \in \operatorname{int} X$. Then, by definition, $x + \varepsilon B_1(0) \in X$ for some $\varepsilon > 0$ and, in particular, $x + \varepsilon e_k \in X$ for all k (where e_k is k-th coordinate vector. Then, we should have

$$\sum_{n \ne k} n^2 x_n^2 + k^2 (x_k + \varepsilon)^2 \le 1$$

for all k which is *impossible* since this sum tends to infinity as $k \to \infty$.

Part 3.

Problem 1. Since V is a subspace of l_∞ and l_∞ is *complelte*, we only need to show that $\bar{V} = c_0$ (then c_0 will be automatically closed). Indeed, let $x^k \in V$ and $x^k \to x^0$ in l_∞. We need to show that

$$\lim_{n\to\infty} x_n^0 = 0.$$

Let $\varepsilon > 0$ be arbitrary. Since $x^k \to x^0$ in l^∞, we may find $k = k(\varepsilon)$ such that

$$|x_n^0| \le |x_n^0 - x_n^k| + |x_n^k| \le \|x^k - x^0\|_{l^\infty} + |x_n^k| \le \varepsilon + |x_n^k|.$$

Finally, since $x^k \in V$, there is $N = N(k(\varepsilon))$ such that $x_n^k = 0$ for $n \ge N$. Thus, $|x_n^0| \le \varepsilon$ if $n \ge N$ and $x^0 \in c_0$.

Let now $x^0 \in c_0$. Construct a sequence $x^k \in V$ by replacing x_n^0 with $n \ge k$ by zeros. We claim that $x^k \to x^0$ in l^∞. Indeed,

$$\|x^k - x^0\|_{l^\infty} = \sup_{n\ge k} |x_n^0| \to 0$$

as $k \to \infty$ (since $\lim_{n\to\infty} x_n = 0$). Thus, any $x^0 \in c_0$ belongs to $\bar{V}$ and we have proved that $\bar{V} = c_0$.

Problem 2. This space is not complete since there exists sequences of C^1-functions which converges in the sup-norm to the non-smooth ones (for instance, ANY continuous function can be approximated in sup-norm by smooth polynomials – Weierstrass theorem).

Problem 3. Unlike the previous example, this space is *complete*. Indeed, let f_n be a Cauchy sequence (in the sup-norm) of analytic functions. Then, since $C(\bar{D})$ is *complete*, it converges in the sup-norm to some *continuous* function f_0. In order to prove the desired completness, we only need to check that f_0 is *analytic* ($f_0 \in H(\bar{D})$). To this end, we take any smooth closed curve Γ and see that

$$\oint_\Gamma f_0(z)\partial\Gamma = \lim_{n\to\infty} \oint_\Gamma f_n(z)\partial\Gamma = 0$$

and f_0 is analytic by the criterium.

Problem 4. Let $x_0 \in \operatorname{int} V$. Then, there is $\varepsilon > 0$ such that $x_0 + \varepsilon \bar{B}_1(0) \subset V$. Since the closed unit ball $\bar{B}_1(0)$ in V is not compact, there exists a sequence $e_n \in B_1(0)$ which does not contain any convergent subsequences. Then, the sequence $x_n := x_0 + \varepsilon e_n \in V$ also does not contain any convergent subsequences which contradicts the compactness of V.

Problem 5. See Exam 2008.

Problem 6.

Proof 1: Let $x^k \in K$ be an arbitrary sequence. Then, the sequence $x_n^k \in \mathbb{R}$ is bounded for any fixed n and, therefore, contains a convergent subsequence. Using the Cantor diagonal procedure (see lectures), we may extract a subsequence x^{k_l}

such that $x_n^{k_l} \to x_n^0$ as $l \to \infty$ (in other words, x^{k_l} is convergent to some $x^0 \in \mathbb{R}^\infty$ coordinate-wise). We need to check that $x^0 \in K$ and that $x^{k_l} \to x^0$ in l^∞ (not only coordinate-wise).

First, since $n|x_n^{k_l}| \le 1$, for all fixed n and all l, passing to the limit $l \to \infty$, we see that $n|x_n^0| \le 1$. This shows that $x^0 \in K$ (and, in particular, K is closed).

Second,

$$\|x^{k_l} - x^0\|_{l^\infty} = \sup_{n\in\mathbb{N}} |x_n^{k^l} - x_n^0| \le \sup_{n\le N} + \sup_{n>N} \le \sup_{n\le N} |x^{k_l} - x_n^0| + \frac{2}{N}.$$

Then, the second term into the right-hand side can be made less than $\varepsilon/2$ by the proper choice of N and the first one tends to zero as $l \to \infty$ for every fixed N (due to the coordinate-wise convergence). Thus, indeed, $x^{k_l} \to x^0$ in l^∞ and K is compact.

Proof II: Analogously to proof I, K is *closed* subspace of complete space l^∞ and, therefore, K is complete. So, by Hausdorff criterium, we only need to check that K is totally bounded. To see that, we introduce the sets

$$K_N := \{x \in l^\infty,\ |x_n| \le 1,\ n \le N;\ x_n = 0,\ n > N\}.$$

Obviously, K_N are closed bounded subsets (cubes) of $\mathbb{R}^N$ and, therefore, they are compact. So, by Hausdorff criterium, every K_N is *totally bounded.* Note now that, since $|x_n| \le 1/n$, for every $x \in K$,

$$K \subset K_N + B_{1/N}(0)$$

and every $1/N$-net of K_N (which exists, see above) will be the $2/N$-net of K. Since N is arbitrary, K is totally bounded.

Problem 7. By the definition of the supremum, there are sequences $x_n, y_n \in K$ such that $\operatorname{diam} K = \lim_{n\to\infty} d(x_n, y_n)$. Since K is compact, there are convergent subsequences $x_{n_k} \to x_0 \in K$ and $y_{n_k} \to y_0 \in K$. Since $d(x, y)$ is continuous in both arguments, passing to the limit $k \to \infty$, we will have $\operatorname{diam} K = d(x_0, y_0)$.

Problem 8. See Exam 2008.

Part 4.

Problem 1. The point-wise limit is $f_0(x)$ which is zero if $x < 1$ and one if $x = 1$. Since the limit function is discontinuous, the convergence cannot be uniform.

Problem 2.

a) $w(z) = z$ (since $||x+z| - |x|| \le |z|$ and the achieved for some x);

b) The function is C^1, so, by the mean value theorem, $\omega(z) = Cz$ with $C = \max_{x\in[0,2\pi]} |f'(x)| = 2$;

c) Particular case of d);

d) Since $f(z) - f(0) = z^p$, $\omega(z) \ge z^p$. Let us prove the equality. Indeed, for any $x, z \in \mathbb{R}_+$, $z \ne 0$ and $x = tz$,

$$\sup_{x,y\in\mathbb{R}_+} \frac{(x+z)^p - x^p}{z^p} = \sup_{t\in\mathbb{R}_+} \{(1+t)^p - t^p\}$$

Let $F(t) := (1+t)^p - t^p$. Then $F'(t) = \frac{p}{(1+t)^{1-p}} - \frac{p}{t^{1-p}} < 0$. Thus, $F(t)$ is decaying and, by this reason, it achieves the supremum at $t = 0$ and this sup is one and

$$(x+z)^p - x^p \le z^p.$$

e) The function is *discontinuous* at $x = 0$ and cannot possess the modulus of continuity.

a)b) – Lipschitz, a)-d) – Hölder.

Problem 3. According to the inequality and the definition of the modulus of continuity, for any $x \ne 0$, we have

$$d(f(x), f(0)) \le \sum_{k=0}^{n-1} \omega(\|1/nx\|) = n\omega(\|x\|/n) = \|x\| \cdot (\|x\|/n)^{-1}\omega(\|x\|/n).$$

Since n is arbitrary here, and the left hand side is independent of n, we may pass to the limit $n \to \infty$ here. Moreover, using that $z^{-1}\omega(z) \to 0$ as $z \to 0$, we see that the right-hand side tends to zero. Thus, $d(f(x), f(0)) = 0$ and $f(x) \equiv f(0)$.

Problem 4. Let $f_N(x)$, $x \in [0,1]$ be a continuous function which equals zero if $x \notin [1/(2N), 1/N]$ and such that $\|f_N\|_{sup} = 1$ (obviously, such functions exist for every N, e.g., piece-wise linear). Then, $f_N \to 0$ point-wise (for every fixed x, only finitely many $f_N(x)$ are not zero), the limit function (equals zero identically) is continuous, but the convergence is not uniform since $\|f_n\|_{sup}$ does not tend to zero.

Problem 5. The initial function f_0 is continuous and equals zero outside of $[-\pi/2, \pi/2]$. Since, $[-\pi, \pi]$ is compact, it must be uniformly continuous on it and, therefore, on $x \in \mathbb{R}$ as well. And, clearly, $\|f_0\|_{sup} = 0$.

The functions f_n are just shifts of the initial function f_0, so $\omega_{f_n}(z) = \omega_{f_0}(z)$ and $\|f_n\|_{sup} = \|f_0\|_{sup}$. Thus, this sequence of functions is uniformly bounded and uniformly continuous.

Since $f_n(x)$ and $f_m(x)$ cannot be non-zero simultaneously, $\|f_n - f_m\|_{sup} = \max\{\|f_n\|_{sup}, \|f_m\|_{sup}\} = 1$. Thus, this sequence does not contain any convergent subsequence. This does not contradict the Arzela theorem, since this theorem gives the compactness criterium for $C[a,b]$ and not for $C(\mathbb{R})$.

Problem 6. Clearly, the set K is closed and uniformly continuous. We only need to check that K is bounded in $C[-1,1]$. We claim that $|f(0)| \le 1$ for every $f \in K$. Indeed, if, say, $f(0) > 1$ then, since $|f(0) - f(x)| \le |x|^{1/2} \le 1$ the function will be sign-defined and cannot have zero mean. After that, it is easy to see that

$$|f(x)| \le |f(0)| + |f(0) - f(x)| \le 1 + |x|^{1/2} \le 2$$

and, therefore, $\|f\|_{sup} \le 2$ for all $f \in K$ and K is bounded. Thus, by Arzela theorem, K is compact.

Part 5.

Problem 1.

a) Due to mean value theorem, $|f(x) - f(y)| = |f'(\xi)||x - y|$ and $|f'(\xi)| = 1/2(\xi+1)^{-1/2} \le 1/2$. So, it is a contraction with $\kappa = 1/2$. The fixed point satisfies $\sqrt{x+1} = x$ or $x = (1+\sqrt{5})/2$.

b) Here, we factually have $F(f) \in \mathbb{R}$ is a number (which, of course, can be considered as a constant function):

$$|F(f_1) - F(f_2)| = |\int_0^1 s(f_1(s) - f_2(s))\,ds| \le$$
$$\le \int_0^1 s|f_1(s) - f_2(s)|\,ds \le \|f_1 - f_2\|_C \int_0^1 s\,ds = 1/2\|f_1 - f_2\|_C$$

and F is a contraction. Clearly, the fixed point is a constant function $f(x) \equiv C$ where C satisfies

$$C = 1 + C\int_0^1 s\,ds = 1 + C/2.$$

Thus, $C = 2$.

c) It is *not* a contraction. Indeed, the fixed point of this map should satisfy $\xi = \sqrt{\xi^2+1}$ or $\xi^2 = \xi^1 + 1$, so there are no fixed points here and, by Banach theorem, it cannot be a contraction.

d) Let $\varphi(z) := \sqrt{1+z}$. Then, as we have seen in a), $|\varphi(z_1) - \varphi(z_2)| \le 1/2|z_1 - z_2$. Let now $f_1, f_2 \in C(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$ be arbitrary. Using the last formula with $z_i = x + f_i(x)$, we have

$$|F(f_1)(x) - F(f_2)(x)| \le 1/2|f_1(x) - f_2(x)|.$$

Taking the supremum over all x, we see that $\|F(f_1) - F(f_2)\|_C \le 1/2\|f_1 - f_2\|_C$, so F is a contraction with the contraction factor 2. To find a fixed point, we need to solve the equation

$$f(x) = \sqrt{1 + x + f(x)} \text{ or } f^2(x) = 1 + x + f(x).$$

Solving this quadratic equation, we find $f(x) = \frac{1+\sqrt{1+4(1+x)}}{2}$.

e) Not a contraction (see Exam 2008).

Problem 2. Indeed,

$$|Fx_1 - Fx_2| = |A^{-1}B(x_1 - x_2)| \le$$
$$\le \|A^{-1}\| \cdot |B(x_1 - x_2| \le \|A^{-1}\|\|B\||x_1 - x_2| = \kappa|x_1 - x_2|$$

with $\kappa := \|A^{-1}\|\|B\| <$. Thus, F is a contraction on $\mathbb{R}^n$ and, therefore, equation $(1 - A^{-1}B)x = h$ is uniquelly solvable for any $h \in \mathbb{R}^n$. From linear algebra, we conclude that $\det(1 + A^{-1}B) \ne 0$. Since $\det A^{-1} \ne 0$, we see that $\det(A + B) = \det A^{-1}\det(1 + A^{-1}B) \ne 0$.

Problem 3. These sequences are generated by iterating the maps $f(x) = \sqrt{x+2}$ and $f(x) = \sqrt{x+3}$ respectively ($x_{n+1} = f(x_n)$, $x_0 = 0$). Both of that maps are *contractions* on $X = \mathbb{R}_+$ with standard metric (see Problem 1), so these sequences must converge to the fixed points of that maps. For a) this fixed point is 2 and for b) it is $\frac{1+\sqrt{13}}{2} \neq 3$.

Problem 4. See Coursework 2(2008).

Problem 5.

a) No, since the right-hand side $f(y) = 3y^{2/3}$ is not *Lipschitz* continuous at $y = 0$.

b) This solution, defined on the interval $[0, \infty)$ is unique since $y(t) \geq 1$ for $t \geq 0$ and the local uniqueness theorem holds near every point $(t, y(t))$, $t \geq 0$ (the function $3y^{2/3}$ is locally Lipschitz if $y \neq 0$.

c) The conditions of the local uniqueness theorem fail at $t = -1$ ($y(-1) = 0$), so the non-uniqueness may appear. It is indeed the case, since the function $y_4(t) = 0$ if $t \leq -1$ and $y_4(t) = (t+1)^3$ for $t \geq 1$ is one more solution of that problem.

Problem 6. We know (Problem 7 Part 3) that $\operatorname{diam}(F(X)) = d(f(x_0), f(y_0))$ for some $x_0, y_0 \in X$. Due to the non-expanding property, we than have

$$\operatorname{diam}(F(X)) < d(x_0, y_0) \leq \operatorname{diam}(X).$$

So, the sequence D_n is monotone decreasing and there is a limit $D_0 := \lim_{n\to\infty} D_n$. Assume that $D_0 > 0$. Let $x_n, y_n \in F^n(X)$ be such that $D_{n+1} = d(f(x_n), f(y_n))$. Then,

$$D_{n+1} = d(f(x_n), f(y_n)) < d(x_n, y_n) \leq D_n$$

and both sides of that inequality converge to D_0, we have

$$D_0 = \lim_{n\to\infty} d(f(x_n), f(y_n)) = \lim_{n\to\infty} d(x_n, y_n).$$

Since X is compact, there are convergent subsequences $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. And the last identity now gives

$$D_0 = d(f(x_0), f(y_0) = d(x_0, y_0).$$

Thus, by the conditions on f, we conclude that $x_0 = y_0$ and $D_0 = 0$.

Let now $x_0 \in X$ and $x_{n+1} := F(x_n)$ be the iterations of the map F. Then, since $X \supset F(X) \supset F^2(X) \supset \cdots$, we know that $x_n, x_{n+k} \in F^n(X)$ and, therefore,

$$d(x_n, x_{n+k}) \leq \operatorname{diam}(F^n(X)) = D_n \to 0$$

as $n \to \infty$. Thus, x_n is a Cauchy sequence in X and, therefore, it is convergent to some $x_0 \in X$ which is the unique fixed point of F.

This extension of Banach contraction theorem does not work for the non-compact case. The counterexamples are the maps c) or e) from Problem 1.

Problem 7. See Coursework 2 (2008).

Part 6.

Problem 1.
a) Yes.
b) No. It is not by-linear.
c) No. It is not positive definite.
d) Yes.
e) No. It is not correctly defined for all $x, y \in l^2$.

Problem 2. We need to prove that $\|xy\|^2 = \sum_{n=1}^{\infty} x_n^2 y_n^2 < \infty$. Using the Cauchy-Schwartz inequality, we have

$$\sum_{n=1}^{\infty} x_n^2 \cdot y_n^2 \leq \left(\sum_{n=1}^{\infty} (x_n^2)^2\right)^{1/2} \left(\sum_{n=1}^{\infty} (y_n^2)^2\right)^{1/2} = \|x\|_{l^4}^2 \|y\|_{l^4}^2 < \infty.$$

(Being pedants, we need to consider first the finite sums from $n = 1$ to N and the pass to the limit $N \to \infty$).

Problem 3.
a) $\pi/2$, since they are orthogonal.
b) $\pi/2$ by the same reason.
c) $(|x|^{-1/3}, |x|^{-1/3}) = 2$, $\||x|^{-1/3}\|_{L^2}^2 = \int_{-1}^{1} x^{-2/3}\, dx = 6$, $\||x|^{2/3}\| = 6/5$ and, therefore,

$$\cos\phi = \frac{2}{\sqrt{6}\sqrt{6/5}} = \frac{\sqrt{5}}{3}$$

and $\phi = \arccos(\sqrt{5}/3)$.

Problem 4. Let us prove that it is continuous at $x_0, y_0 \in H$. Indeed, let $x \in H$ and $y \in H$ be close to x_0 and y_0 respectively (in particular, $\|x - x_0\| \leq 1$ and $\|y - y_0\| \leq 1$). Then, due to Cauchy-Schwartz inequality,

$$|(x_0, y_0) - (x, y)| = |(y_0, x_0 - x) + (x, y_0 - y)| \leq |(y_0, x_0 - x)| + |(x, y_0 - y)| \leq$$
$$\leq \|y_0\|\|x_0 - x\| + (\|x_0\| + 1)\|y_0 - y\| \leq (\|y_0\| + \|x_0\| + 1)(\|x - x_0\| + \|y - y_0\|)$$

which gives the continuity.

Problem 5. $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2(x, y)$.

Problem 6. See Coursework 1 (year 2007) for the module "Functional Analysis and PDEs" (solutions are on my web-page).

Part 7.

Problem 1. a) Clearly, H_0 is a closed linear subspace of H, so H_0 is also a Hilbert space and $\{e_n\}$ is an orthonormal system in it. In order to check that it is a basis, we only need to verify the *completness*. Let $y_0 \in H$ be such that (y_0, e_n) for all n. Then, $(y_0, x) = 0$ for all x which are *finite* linear combinations of e_n. Since such combinations are dense in H_0 (by definition), we have that $(y_0, x) = 0$ for all $x \in H_0$. In particular, for $x = y_0$. Thus $y_0 = 0$ and $\{e_n\}$ is complete in H_0.

b) Let $x \in H$ and $P : H \to H_0$ is defined by $Px = x_0 := \sum_{n=1}^{\infty}(x, e_n)e_n$ (we know from lectures that the series is convergent. Then P is *linear* : $P(\alpha x + \beta y) = \alpha Px + \beta Py$; $P^2 x = Px_0 = x_0 = Px$ (since $(x, e_n) = (x_0, e_n)$) and $x - Px = x - x_0$ is orthogonal to H_0. Thus, P is an orhtoprojector from H on H_0.

Problem2. A countable dense set is given, e.g., by all *finite* linear combinations of $\{e_n\}$ with *rational* coefficients.

Problem 3. Direct calculations: general formulas for Legendre and Laguerre polynomials as well as list of first polynomials can be found, say, in Wikipedia.

Problem 4.

a) Follows, e.g., from the explicit formulas for the coefficients a_n and b_n.

b) Integrate by parts k-times in the formulas for the coefficients and use the Bessel inequality for $f^{(k)}$.

Problem 5.

a) First: $f(x) = x$. It is odd, so only b_n's are non-zero:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = -\frac{1}{\pi n} x \cos nx\Big|_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi} \cos nx \, dx = -2\frac{(-1)^n}{n}.$$

Thus,

$$x \sim -2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

Second: $f(x) = |x|$. It is even, so all b_n are zero. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \, dx = \pi/2$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi n} x \sin nx\Big|_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} \sin nx \, dx = -2\frac{1-(-1)^n}{2\pi n^2}.$$

Thus,

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$$

Third: $f(x) = e^x$. This function is neither even nor odd, so all coefficients must be computed. To do that, it is useful to use $e^{inx} = \cos nx + i \sin nx$. Indeed,

$$\int_{-\pi}^{\pi} e^x e^{inx} \, dx = \frac{1}{1+in} e^{(1+in)x}\Big|_{-\pi}^{\pi} =$$

$$= \frac{1}{1+in}(-1)^n\left(e^{\pi} - e^{-\pi}\right) = 2(-1)^n \sinh \pi \frac{1}{1+ni} = \frac{2(-1)^n \sinh \pi}{1+n^2}(1-ni).$$

Therefore, $a_0 = \frac{\sinh\pi}{\pi}$,

$$a_n = \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}, \quad b_n = -\frac{2(-1)^n n \sinh\pi}{1+n^2}$$

and

$$e^x \sim \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx - \frac{(-1)^n n}{1+n^2}\sin nx.$$

b) All the above functions are point-wise smooth, so the Dirichlet theorem works. In all 3 cases the expansions converge point-wise to the proper periodic extensions in the points of continuity and to the mean points at the jump points. The periodic extensions for first and third functions have jumps, so the convergence is not uniform. The second function does not have jumps and the convergence is uniform (the periodic extension is continuous and piece-wise C^1).

c) Since $\cosh x = \frac{e^x+e^{-x}}{2}$, due to formulas of Problem 3, we have

$$\cosh x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos nx$$

(no jumps for periodic extension => the convergence is uniform).

Since $\sinh x = \frac{e^x - e{-x}}{2}$,

$$\sinh x = -\frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{1+n^2}\sin nx$$

(except of jump points $x = \pm\pi$ where we have convergence to zero).

Finally, $x_+ = (x+|x|)/2$, so using the found expansions for x and $|x|$, we find

$$x_+ = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

(except of the jump points $x = \pm\pi$ where we have convergence to $\pi/2$).

d) Taking $x = 0$ in the expansions for $|x|$, we find

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \quad => \quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Let now $Z = \sum_{n=1}^{\infty}\frac{1}{n^2}$. Then,

$$Z = \sum_{n=2k}\frac{1}{n^2} + \sum_{n=2k+1}\frac{1}{n^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{\pi^2}{8} = \frac{Z}{4} + \frac{\pi^2}{8}$$

and $Z = \frac{\pi^2}{6}$.

Finally, for computing $\sum_{n=0}^{\infty}\frac{1}{n^2+1}$, put $x = \pi$ into the expansions of $\cosh x$. Then,

$$\cosh\pi = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

which gives

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2\tanh\pi} + \frac{1}{2}.$$

(do not forget to add one for the term with $n = 0!$).

e) We use that $\|f\|_{L^2}^2 = 2\pi|a_0|^2 + \pi \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ (Parseval equality).

For the case $f(x) = |x|$, $\|f\|_{L^2}^2 = \frac{2}{3}\pi^3$, so

$$\frac{2}{3}\pi^3 = \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}$$

and $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$.

Let now $Z = \sum_{n=1}^{\infty} \frac{1}{n^4}$. Then, as before,

$$Z = \sum_{n=2k} \frac{1}{n^4} + \sum_{n=2k+1} \frac{1}{n^4} = \frac{Z}{16} + \frac{\pi^4}{96}$$

and $Z = \frac{\pi^4}{90}$.

Finally, let $f(x) = \cosh x$. Then, $\|f\|_{L^2}^2 = \pi + \frac{1}{2}\sinh 2\pi$ and

$$\pi + \frac{1}{2}\sinh 2\pi = \frac{2\sinh^2\pi}{\pi} + \frac{4\sinh^2\pi}{\pi}\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2}$$

which gives

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^2} = \frac{2\pi^2 + \pi\sinh 2\pi + 4\sinh^2\pi}{8\sinh^2\pi}.$$

(of course, if I am not mistaken! If anybody find a mistake, please, let me know!!!)