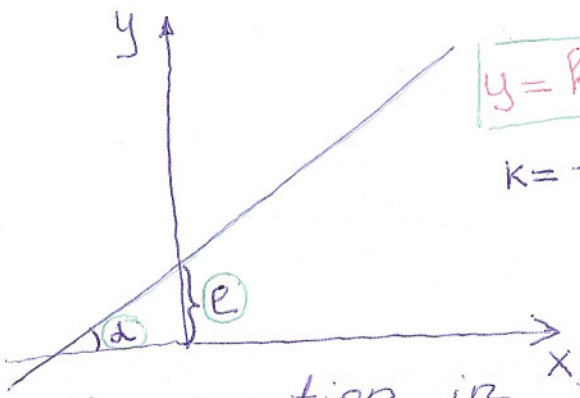
Lecture: Vectors, lines and planes in $\mathbb{R}^3$

Aims:

- Remind the properties of dot, cross and box products
- learn different ways to present lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$
- understand how to use dot and cross products for computing various quantities related with lines and planes.

Start with $\mathbb{R}^2$:



$y = kx + \ell$

$k = \operatorname{tg}\alpha$

Line equation in the

1) explicit form: $y = kx + \ell$

(solved w.r.t. y, $y = f(x)$)

Drawback: gives not all lines
- vertical lines cannot be expressed in that form.

2) Implicit form: $f(x, y) = 0$

$$ax + by + c = 0$$

a, b, c - given parameters

vertical lines correspond to $b = 0$

3) Parametric form $x = x(t),\ y = y(t)$

$$x = x_0 + \alpha t$$
$$y = y_0 + \beta t$$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ are given

t-parameter

$t \in \mathbb{R}$

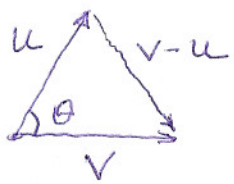
Vectors: $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

dot product:

$$u \cdot v := u_1 v_1 + u_2 v_2$$

$|u|^2 = u \cdot u$

cos-rule:



$$|v-u|^2 = |u|^2 + |v|^2 - 2|u|\cdot|v|\cdot\cos\theta$$

$$\| \quad (v-u)\cdot(v-u) = |u|^2 + |v|^2 - 2u\cdot v$$

$$\Rightarrow \quad u \cdot v = |u|\cdot|v|\cos\theta$$

$$u \perp v \iff u \cdot v = 0$$

u is orthogonal to v

Angle between two vectors

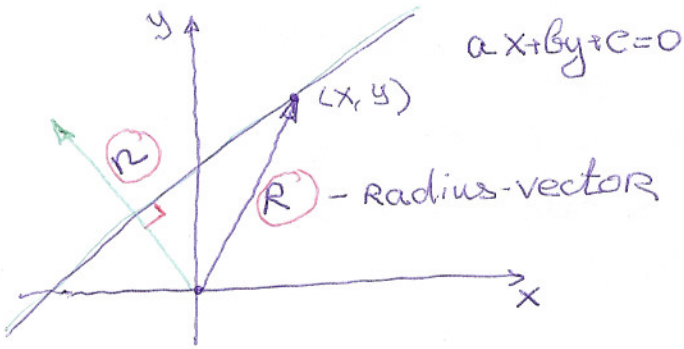
$$\cos\theta = \frac{u\cdot v}{|u|\cdot|v|}$$

$$\theta = \cos^{-1}\left(\frac{u\cdot v}{|u|\cdot|v|}\right)$$

Geometric interpretation:

Implicit form $\boxed{aX+bY+c=0}$

Introduce: $R=\begin{pmatrix}x\\y\end{pmatrix}$, $n=\begin{pmatrix}a\\b\end{pmatrix}$



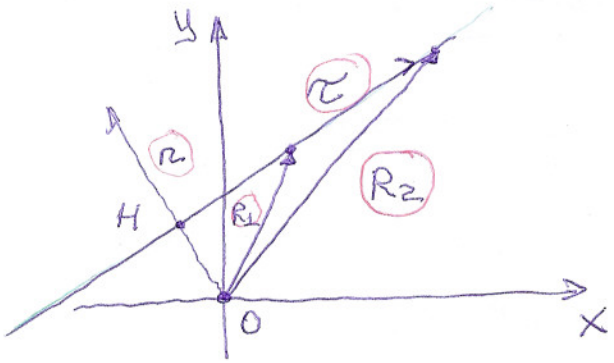
$aX+bY+c=0 \iff \boxed{n\cdot R=-c}$ for all x, y

What does it mean?

Take any 2 points $R_1=\begin{pmatrix}x_1\\y_1\end{pmatrix}$ and $R_2=\begin{pmatrix}x_2\\y_2\end{pmatrix}$ on the line

then $\boxed{\tau = R_2-R_1}$

belongs to the line



$\boxed{n\cdot\tau = n\cdot R_2 - n\cdot R_1 = -c-(-c)=0}$

$\boxed{n\perp\tau}$ $\Rightarrow$ Vector $\boxed{n=\begin{pmatrix}a\\b\end{pmatrix}}$ is orthogonal to the line $\boxed{aX+bY+c=0}$ $\Rightarrow$ geometric sense of coefficients a and b: $n=\begin{pmatrix}a\\b\end{pmatrix}$ is a normal vector to the line.

What is c? measures the distance between the line and the origin. Indeed $D=|\overrightarrow{OH}|$

$$-c=\overrightarrow{OH}\cdot n=|\overrightarrow{OH}|\cdot|\overrightarrow{n}|\underbrace{\cos\theta}_{\pm 1} \qquad \boxed{D=\frac{|c|}{\sqrt{a^2+b^2}}}$$

Typical problems:

1) Find the equation of line passing through $\begin{pmatrix}x_0\\y_0\end{pmatrix}$ and orthogonal to $n=\begin{pmatrix}a\\b\end{pmatrix}$

Answer: $\boxed{a(X-X_0)+b(Y-Y_0)=0}$

2) $-\|-$ and parallel to $\boxed{\tau=\begin{pmatrix}\alpha\\\beta\end{pmatrix}}$

Answer: $\boxed{-\beta(X-X_0)+\alpha(Y-Y_0)=0}$

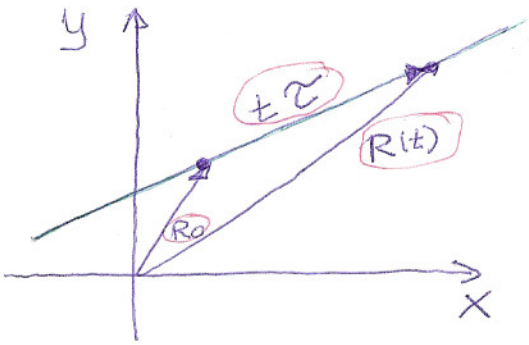
Indeed $n=\begin{pmatrix}-\beta\\\alpha\end{pmatrix}\perp\tau$ $\quad n\cdot\tau=-\alpha\beta+\alpha\beta=0.$

3) Find the angle between lines $\boxed{a_1X+b_1Y+c_1=0}$ and $\boxed{a_2X+b_2Y+c_2=0}$

Answer: $\boxed{\cos\theta=\frac{n_1\cdot n_2}{|n_1|\cdot|n_2|}}$, $\boxed{n_i=\begin{pmatrix}a_i\\b_i\end{pmatrix}}$

Geometric interpretation:

Parametric form:



$$\begin{cases} x = x_0 + \alpha t \\ y = y_0 + \beta t \end{cases}$$

Introduce $R = \begin{pmatrix} x \\ y \end{pmatrix}$, $R_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\tau = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Then

$$R(t) = R_0 + t\tau$$

Describes constant speed motion along the line starting from (x_0, y_0) with velocity τ.

How to transform parametric to implicit?

Way I: Exclude parameter t directly. Multiply 1st and 2nd equations on $-\beta$ and α and take sum

$$-\beta x + \alpha y = -\beta x_0 + \alpha y_0 - \alpha\beta t + \alpha\beta t \Rightarrow -\beta(x - x_0) + \alpha(y - y_0) = 0$$

Way II: Use that $n = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \perp \tau = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and (x_0, y_0) belongs to the line $\Rightarrow$ $-\beta(x - x_0) + \alpha(y - y_0) = 0$

implicit to parametric:

Example: Write down the equations of motion along the line $3x + 4y = 7$ with the constant speed $|V| = 2$ starting from $(1, 1)$.

Solution: $n = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \tau = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$. Normalize the speed $V = 2\frac{\tau}{|\tau|} = \frac{2}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Answer:

$$R(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{5} t \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\begin{cases} x = 1 - \frac{8}{5} t \\ y = 1 + 6/5\, t \end{cases}$$

Warning: The solution

$$\begin{cases} x = 1 + \frac{8}{5} t \\ y = 1 - \frac{6}{5} t \end{cases}$$

which corresponds to the motion in opposite direction $t \to -t$

is also correct.

The 3d case ($\mathbb{R}^3$)

There 3 operations with vectors $u = (u_1, u_2, u_3)^t$

1) Dot (scalar) product : $\boxed{u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3}$

(dot product of two vectors is a scalar $\in \mathbb{R}$)

2) Cross (vector) product :

$$\boxed{u \times v = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - v_3 u_1,\ u_1 v_2 - u_2 v_1)^t}$$

OR

$$\boxed{u \times v = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}$$

e_1, e_2, e_3 – coordinate ords (the same as i, j, k)

(cross product of two vectors is a vector)

3) Box (triple) product :

$$\boxed{[u, v, w] = \det \begin{vmatrix} w_1, & w_2, & w_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix}}$$

(Box product of three vectors is a scalar $\in \mathbb{R}$)

By the det - properties

$$[u, v, w] = w_1 \begin{vmatrix} u_2, & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1, & v_2 \end{vmatrix} = (u \times v) \cdot w$$

$\Rightarrow$

$$\boxed{[u, v, w] = (u \times v) \cdot w = (v \times w) \cdot u = (w \times u) \cdot v}$$

Geometric interpretation :

1) $\boxed{u \cdot v = |u| \cdot |v| \cos\theta}$ – exactly as in 2D

allows us to compute angles

2) $u \times v$ is orthogonal to both u and v

$\boxed{|u \times v| = |u| \cdot |v| \cdot \sin\theta}$ = "area of parallelogram spanned by u and v

$S = |u \times v|$

Why ?

$(u \times v) \cdot u = [u, v, u] = 0 \Rightarrow \boxed{u \times v \perp u}$

$(u \times v) \cdot v = [u, v, v] = 0 \Rightarrow \boxed{u \times v \perp v}$

$\Rightarrow u \times v$ is orthogonal to both u and v

To find $|u \times v|$ we use Lagrange's identity

$$|u \times v|^2 = |u|^2 \cdot |v|^2 - (u \cdot v)^2$$

OR in coordinates

$$(u_1v_2 - u_2v_1)^2 + (u_2v_3 - u_3v_2)^2 + (u_3v_1 - v_3u_1)^2 = (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2$$

expand brackets and check that is true OR see Algebra notes from previous semester.

From Lagrange $\Rightarrow$

$$|u \times v|^2 = |u|^2 \cdot |v|^2 - |u|^2|v|^2\cos^2\theta = |u|^2 \cdot |v|^2(1-\cos^2\theta) = |u|^2 \cdot |v|^2 \sin^2\theta.$$

W How to find the direction of $u \times v$?
(distinguish between two opposite directions)

Answer: By the right hand rule
$(u, v, u \times v)$ form positively oriented bases
$u \times v$ points the way your right thumb points when your fingers curl from u to v.

In particular, cross product is useful for
- finding the normal direction to the plane generated by two vectors
- finding the areas of parallelograms, and triangles in $\mathbb{R}^3$.

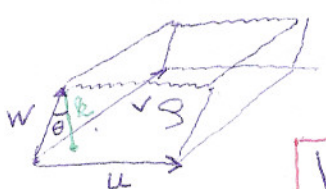
Application to $\mathbb{R}^2$: We may think about the 2D vectors as 3D ones with 0 3rd component

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \sim \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$ then $u \times v = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3$

$\Rightarrow$ area of the parallelogram spanned by 2D vectors u and v

$$S = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

3) Box product: $[u, v, w] = V$ - signed volume of the parallelepiped spanned by u, v and w (sign is + if u, v, w are positively oriented and - otherwise)



Why? From geometry $V = S \cdot h$ where S is the area of the origin. But $S = |u \times v|$
$h = |w|\cos\theta$, h is parallel to $u \times v$

$$V = Sh = |w| \cdot |u \times v| \cdot \cos\theta = (u \times v) \cdot w = [u, v, w]$$

Useful, in particular, for computing volumes!

Some interesting relations:

1) $\boxed{u\times(v\times w)=(u.w)v-(u,v)w}$

2) $\boxed{u\times(v\times w)+v\times(u\times v)+w\times(u\times v)=0}$

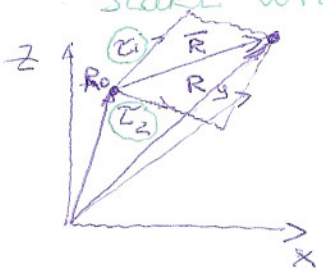
3) $\boxed{(u\times v)\times(u\times w)=[u,v,w]\,u}$

4) $\boxed{(u\times v).(w\times z)=(u.v)(v.z)-(u.z)(v.w)}$

Good training to prove them as exercises!!!

Planes and lines in $\mathbb{R}^3$:

Start with planes:



1) Parametric representation:

$\boxed{R=\begin{pmatrix}x\\y\\z\end{pmatrix}}$ - Radius vector of an arbitrary point on a given plane,

$\boxed{R_0=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}}$ - fixed point on a plane

$\boxed{\tau_1=\begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\end{pmatrix}}$ and $\boxed{\tau_2=\begin{pmatrix}\alpha_2\\\beta_2\\\gamma_2\end{pmatrix}}$ - two non-collinear vectors belonging to the plane

Than Any vector on a plane is a linear combination of τ_1 and τ_2 (can be considered as a def of a plane)

Thus, $\boxed{\overline{R}=R-R_0=s\tau_1+t\tau_2}$ for some $(t,s)\in\mathbb{R}^2$

This gives the desired parametric equations

$$\boxed{R(t,s)=R_0+s\tau_1+t\tau_2}$$

OR in coordinates

$$(P)\quad\boxed{\begin{aligned}x&=x_0+\alpha_1 s+\alpha_2 t\\y&=y_0+\beta_1 s+\beta_2 t\\z&=z_0+\gamma_1 s+\gamma_2 t\end{aligned}}\qquad\begin{matrix}t\in\mathbb{R}\\s\in\mathbb{R}\end{matrix}$$

Warning: different τ_1 τ_2 and R_0 may parametrize the same plane!!!

2) Implicit parameter form ($f(x,y,z)=0$)

may be obtained excluding the parameters (t) and (s) from (P) using Gaussian elimination (works $\Leftrightarrow$ τ_1 and τ_2 not collinear)

$\Rightarrow$ $\boxed{ax+by+cz+d=0}$ (i)

desired equation of a plane in 3D

Introducing $\boxed{n=\begin{pmatrix}a\\b\\c\end{pmatrix}}$ we write (i) via the dot product

$$\boxed{n.R=-d}$$

and see (exactly as in the 2D case) that n is orthogonal to any vector on the plane (n is a normal vector to the plane)

$\Rightarrow \boxed{n\perp\tau_1}$ and $\boxed{n\perp\tau_2} \Rightarrow \boxed{n=\tau_1\times\tau_2}$ - defined up to a scalar multiplier

This gives a simple way to find (i) if (p) is given:

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0} \qquad \boxed{n=\begin{pmatrix}a\\b\\c\end{pmatrix}=\tau_1\times\tau_2}$$

Example: The plane is given by

$$\boxed{x=3+2s+5t,\ y=1+s+t,\ z=-1+s-t}$$

Write down the equation in the (i) form.

Solution:

$$\tau_1=\begin{pmatrix}2\\1\\1\end{pmatrix},\ \tau_2=\begin{pmatrix}5\\1\\-1\end{pmatrix} \qquad \tau_1\times\tau_2=\begin{vmatrix}e_1&e_2&e_3\\2&1&1\\5&1&-1\end{vmatrix}=-2e_1+7e_2-3e_3$$

$$\Rightarrow n=\begin{pmatrix}-2\\7\\-3\end{pmatrix}$$

$$-2(x-3)+7(y-1)-3(z+1)=0$$

$$\boxed{2x-7y+3z+2=0}$$

Analogously to 2D case, the coefficient d measures distance between the plane and the origin

$$\boxed{D=\frac{|d|}{\sqrt{a^2+b^2+c^2}}}$$

Exercise: Prove that the distance between plane $ax+by+cz+d=0$ and a point $(x_0,y_0,z_0)\in\mathbb{R}^3$ is

$$\boxed{D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}$$

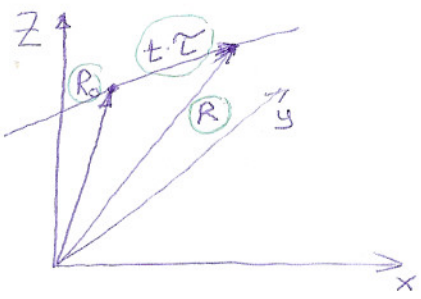
Hint: do parallel shift which moves (x_0,y_0,z_0) to the origin.

Problem: Find the angle between two planes $\boxed{a_1x+b_1y+c_1z+d_1=0}$ and $\boxed{a_2x+b_2y+c_2z+d_2=0}$

Answer: Angle between planes = angle between their normals: $\boxed{\theta=\cos^{-1}\left(\frac{n_1\cdot n_2}{|n_1|\cdot|n_2|}\right)}$

Lines in space:

1) Parametric form



$R_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ - fixed point on a line

$R = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ - arbitrary point, $\tau = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ - vector belonging to the line

$\Rightarrow$ any other vector on a line is collinear to τ

$\Rightarrow R - R_0 = t\tau$, $t \in \mathbb{R}$ $\quad \boxed{R(t) = R_0 + t\tau}$ OR in coordinates

$$\boxed{x = x_0 + \alpha t,\ y = y_0 + \beta t,\ z = z_0 + \gamma t,\ t \in \mathbb{R}}$$

OR one more equivalent form

$$\boxed{\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma}} \quad = t \text{ is excluded here}$$

2) Implicit form: line as an intersection of 2 planes

$$\boxed{\begin{cases} a_1 x + b_1 y + c_1 z + d_1 = 0 \\ a_2 x + b_2 y + c_2 z + d_2 = 0 \end{cases}} \text{ - simultaneous equations}$$

planes should not be parallel: $n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \not\parallel n_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$

Example: The line is given implicitly:

$$\boxed{x + y + z + 5 = 0,\ 2x + y - z + 3 = 0}$$

write it in a parametric form

Solution: vector τ belongs to both planes $\Rightarrow \boxed{\tau \perp n_1}$ and $\boxed{\tau \perp n_2}$

$\Rightarrow \tau = n_1 \times n_2$ $\quad \tau = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2e_1 + 3e_2 - e_3 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$

we need also one point on the line. Seek for it, say, with $\boxed{z = 0}$

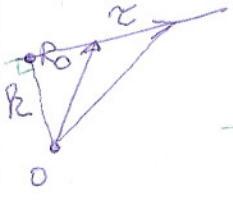
$$\begin{cases} x + y + 5 = 0 \\ 2x + y + 3 = 0 \end{cases} \Rightarrow (x_0, y_0, z_0) = (-2, 7, 0)$$

$$\boxed{x = -2 - 2t,\ y = 7 + 3t,\ z = -t} \iff \boxed{\frac{x+2}{-2} = \frac{y-7}{3} = \frac{z}{-1}}$$

Example: Find the distance between the origin and the line

$$\boxed{x = 2 + t,\ y = 3 + 2t,\ z = 1 + t}$$

Solution. Let S be the area of a triangle spanned by R_0 and τ



Then $\boxed{S = \frac{1}{2}|\tau| \cdot h} = \boxed{\frac{1}{2}|R_0 \times \tau|} \Rightarrow \boxed{h = \frac{|R_0 \times \tau|}{|\tau|}}$

$R_0 = (2, 3, 1)$, $\tau = (1, 2, 1)$

$$R_0 \times \tau = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = e_1 - e_2 + e_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad |R_0 \times \tau| = \sqrt{3}, \quad |\tau| = \sqrt{6}$$

$$\boxed{h = \sqrt{3}/\sqrt{6} = 1/\sqrt{2}}$$