Lecture 2: Differentiable functions of many variables.

Aim: to understand what does it mean that function f is differentiable and to learn how to compute its differential

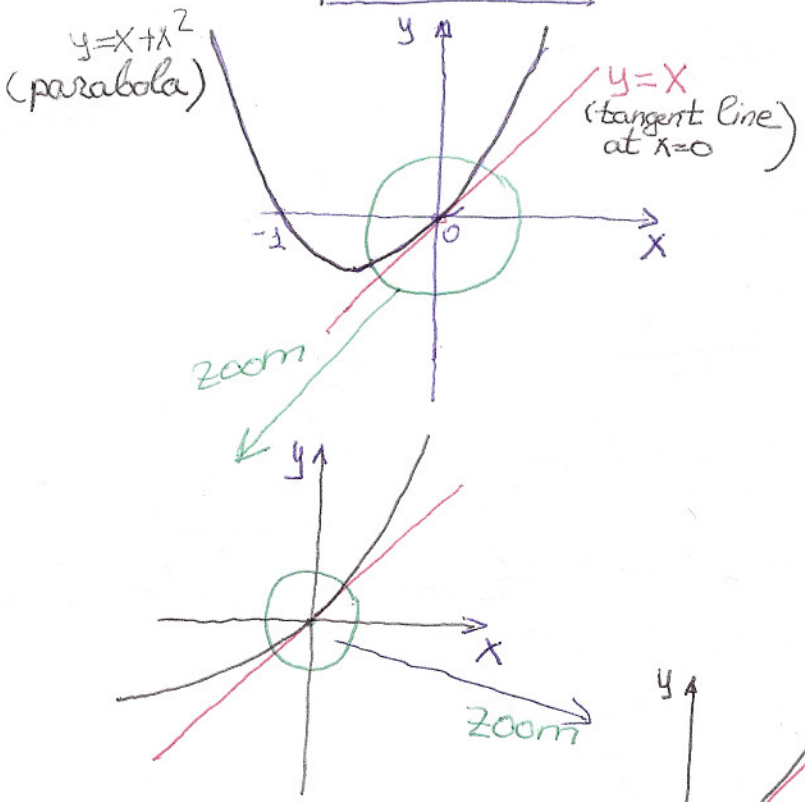
We start with the case of 1-variable

(Reminder)

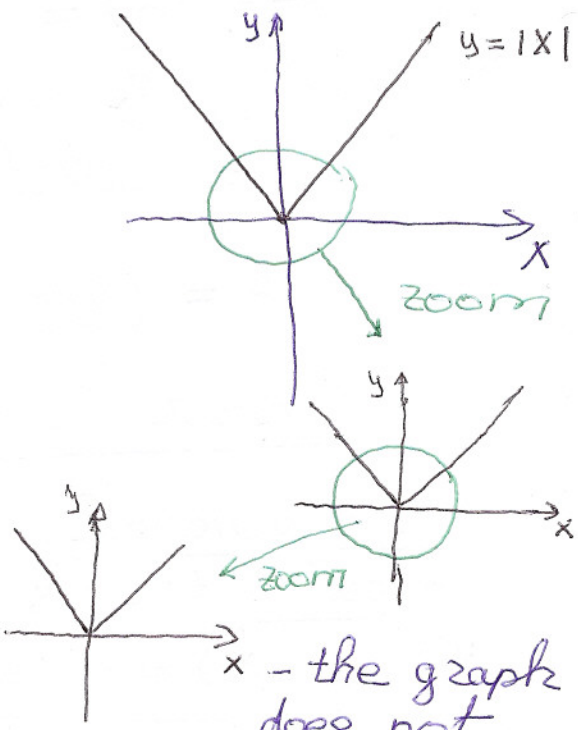
$\boxed{y = f(x),\ x \in \mathbb{R}}$ - what does it mean if f is differentiable, say, at $\boxed{x=0}$?

Consider 2 examples:

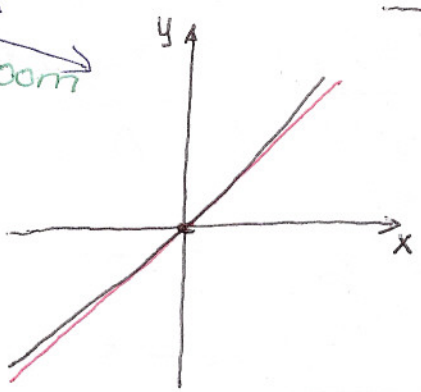
"Good" $\boxed{y = x + x^2}$



"Bad" $y = |x|$



⇒ Under zoom the graph becomes closer and closer to the graph of the Linear function $\boxed{y = x}$



why?

$y(x) = x + x^2$

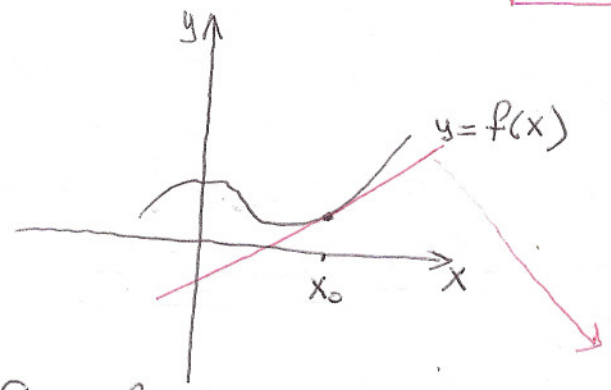
Linear + higher order deviation from linear.

- the graph does not change!

⇒ not differentiable at $x=0$!

General 1D case:

Consider smooth $y = f(x)$ near $x = x_0$



Taylor expansion:

$$y(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{Linear}} + \underbrace{O((x-x_0)^2)}_{\text{Higher order terms (deviation from linear)}}$$

$$y = f(x_0) + f'(x_0)(x-x_0)$$

tangent line at $x = x_0$.

Conclusion:

Differentiable = Linear + H.O.T

Indeed if

$$f(x) - f(x_0) = A(x-x_0) + \underbrace{O((x-x_0)^2)}_{\text{H.O.T.}}$$ for some A

$\Rightarrow$ dividing by $x - x_0$

$$A = \frac{f(x)-f(x_0)}{x-x_0} + O(x-x_0) \Rightarrow$$

$$A = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} := f'(x_0)$$

$\Rightarrow$ A must be the derivative of f at x_0.

More general: if we want to approximate f up to cubic terms near x_0, we write

(*) $$f(x) = f(x_0) + A(x-x_0) + B(x-x_0)^2 + O((x-x_0)^3)$$

We already know that A must be $f'(x_0)$

differentiating (*) by x

$$f'(x) = f'(x_0) + B \cdot 2(x-x_0) + O((x-x_0)^2)$$

$\Rightarrow$ $B = f''(x_0)/2$ - second term in Taylor expansions.

Analogously, approximation of f by polynomial of n-th power (up to order $n+1$) lead to Taylor expansions

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \text{H.O.T}$$

We will use this idea for finding Taylor expansions for many variables.

Infinitesimal increments and differentials:

$y = f(x)$ · consider two points

x_0 and $x_0 + \delta x$ — increment of x (small)

$f(x_0)$ and $f(x_0 + \delta x)$

$\delta y = \delta f = f(x_0 + \delta x) - f(x_0)$ — increment of y

for small δx, we may write

$$\delta y = f'(x_0)\delta x + \underbrace{O(\delta x^2)}_{\text{higher order terms}}$$

Thus,

$$\boxed{\delta y \approx f'(x_0)\delta x}$$

smaller $\delta x \Rightarrow$ better approximation for the infinitesimal increment (= differentials)

$\Rightarrow$

$$\boxed{dy = f'(x_0)\,dx}$$

$\Rightarrow$ Leibnitz notation for the derivative

$$\boxed{f'(x) = \frac{df}{dx}}$$

Case of many variables

(for simplicity many = 2)

$\boxed{z = f(x, y)}$, $x, y \in \mathbb{R}$ — two independent variables

$z \in \mathbb{R}$ — dependent variable

Geometrically: surface in the 3D space (x, y, z).

Main question: what is the derivative / differential of f?

Start with example:

$$z = f(x,y) = \frac{1}{1-x-y}$$

near $(x,y)=(0,0)$ the origin

We know that

$$\frac{1}{1-\xi} = 1 + \xi + \xi^2 + \dots$$ - geometric progression

use $\xi = x+y$

$$z = \underbrace{1 + x + y}_{\text{Linear terms}} + \underbrace{(x+y)^2 + \dots}_{\text{H.O.T (quadratic + qubic, etc.)}}$$

$z = 1 + x + y$ - linear part = tangent plane at the origin

$(x+y)^2 + (x+y)^3 + \dots$ = Higher order deviation from Linear

Conclusion:

As in 1D case:

Definition:

Differentiable function = Linear + H.O.T

Main difference: Linear function of two variables (plane in the (x,y,z) space passing through the origin) has two coefficients

$$z = Ax + By$$

(which correspond to increments of x and y respectively)

Thus, for f - differentiable at (x_0, y_0) we expect

$$\delta z = \delta f = f(x,y) - f(x_0,y_0) = \underbrace{A(x-x_0) + B(y-y_0)}_{\text{Linear}} + \underbrace{\text{quadratic terms of } x-x_0,\ y-y_0 + \text{etc.}}_{\text{H.O.T.}}$$

$\delta x = x - x_0$ - increment of x

$\delta y = y - x_0$ - increment of y

Two independent variables (x,y) $\Leftrightarrow$ two independent increments $(\delta x, \delta y)$ $\Leftrightarrow$ vector of increments $\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$

$$\delta f = A\,\delta x + B\,\delta y + H.O.T$$

Problem: How to find A and B?

Solution: Let us consider **only** the increments of $x \to \delta x$ and $\delta y \equiv 0$

(particular case of general increment)

$x_0 \to x = x_0 + \delta x$ $\quad$ $y = y_0 + 0 = y_0$

$y = y_0$ - is not changing (should be treated as a constant)

$$\delta f = f(x_0 + \delta x, y_0) - f(x_0, y_0)$$

increment for 1-variable function

$x \to f(x, y_0)$ $\quad$ $y = y_0$ is treated as a constant

and

$$\delta f = A\,\delta x + H.O.T$$

$\Rightarrow$ A = derivative of this 1-variable function

$$A = \lim_{x \to x_0} \frac{f(x_0 + \delta x, y_0) - f(x_0, y_0)}{\delta x}$$

This derivative has a **spetial name** = **Partial derivative** of f with respect to x at x_0

and spetial notation:

$$A = f'_x(x_0, y_0) = \partial_x f = \frac{\partial f}{\partial x}$$

Equivalent notations: the derivative is taken with respect to variable x. Variable y is treated as a **constant**!

Analogously: we may do increments only in y-variable:

$$\delta x \equiv 0, \quad y_0 \to y = y_0 + \delta y$$

Then

$$\delta f = f(x_0, y_0 + \delta y) - f(x_0, y_0)$$

and

$$\delta f = B\,\delta y + H.O.T$$

$\Rightarrow$

B = "derivative of 1-variable function $y \to f(x_0, y)$"

$$B = \lim_{y \to y_0} \frac{f(x_0, y_0 + \delta y) - f(x_0, y_0)}{\delta y}$$

x is now constant, but y varies = partial derivative with respect to y.

$$B = f'_y = \partial_y f = \frac{\partial f}{\partial y}$$

Equivalent notations for partial derivative with respect to y.

General increments:

$$\delta f = f'_x \delta x + f'_y \delta y + H.O.T$$

For infinitesimal increments = differentials

$$df = f'_x dx + f'_y dy = (f'_x, f'_y) \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}$$

↓ full differential

vector!!! analog of a derivative

Conclusion: The analog of a derivative for many variables is NOT a scalar, but Vector of Partial derivatives (f'_x, f'_y).

This vector has a special name - GRADIENT and special notation

$$\nabla f = (f'_x, f'_y)$$