Lecture 3: Differentiable functions of many variables (continued)

Aim: - To learn how to compute partial derivatives.
- To clarify the geometric meaning of the gradient.

Start with examples:

1) $z = x \cdot y$

variable; treated as constant

$$\frac{\partial z}{\partial x} = \frac{\partial (x y)}{\partial x} = y \cdot \frac{\partial x}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = \frac{\partial (x y)}{\partial y} = x \frac{\partial y}{\partial y} = x$$

$$\Rightarrow \boxed{dz = y\,dx + x\,dy}$$

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2) $$z = \frac{1}{1-x-y}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{(1-x-y)^2} \cdot \partial_x (1-x-y) = \frac{1}{(1-x-y)^2}$$

chain rule is used for the function of one variable x (y is a constant)

$$\frac{\partial z}{\partial y} = -\frac{1}{(1-x-y)^2} \cdot \partial_y (1-x-y) = \frac{1}{(1-x-y)^2}$$

again chain rule, but now for the function of one variable y (x is constant)

$$\boxed{dz = \frac{dx+dy}{(1-x-y)^2}}$$

3) More complicated function:

$$z = e^{\sin(x \cdot y)}$$

$$\frac{\partial z}{\partial x} = e^{\sin(x \cdot y)} \cdot \frac{\partial}{\partial x}[\sin(xy)] = e^{\sin(xy)} \cdot y \cos(xy)$$

$$\frac{\partial z}{\partial y} = e^{\sin(xy)} \cdot \frac{\partial}{\partial y}[\sin(xy)] = e^{\sin(xy)} \cdot x \cdot \cos(xy)$$

$$\boxed{dz = e^{\sin(xy)} \cos(xy) \cdot (y\,dx + x\,dy)}$$

Conclusion: Partial derivatives are computed Exactly in the same way as derivatives of functions of one variable. Other variables are treated as constants.

Case of functions of more than 2 variables:

say, 3 variables

$f = f(x, y, z)$

dependent variable

3 independent variables

Analogously:

$$\delta f = f'_x \delta x + f'_y \delta y + f'_z \delta z + H.O.T$$

and

$$df = f'_x dx + f'_y dy + f'_z dz$$

and now the gradient is a 3-vector

$$\nabla f = (f'_x, f'_y, f'_z)$$

Example:

$f = x \cdot y \cdot z$

$\frac{\partial f}{\partial x} = yz$ now y, z are constants, x is variable

$\frac{\partial f}{\partial y} = xz$, $\frac{\partial f}{\partial z} = xy$

$$df = yz\,dx + xz\,dy + xy\,dz$$

Case of n-variables:

$f = f(x_1, \dots, x_n)$

dependent variable

n independent variables

$$df = f'_{x_1} dx_1 + \cdots + f'_{x_n} dx_n$$

Gradient

$$\nabla f = (f'_{x_1}, \dots, f'_{x_n})$$

is now n-vector.

with the same rules for computing partial derivatives:

for $\frac{\partial f}{\partial x_i}$ x_i is variable

x_k, $k \neq i$ are constants.

Directional derivatives and gradient:

$z = f(x,y)$

$$\delta z = f_x' \delta x + f_y' \delta y + H.O.T$$
$$dz = f_x' dx + f_y' dy$$

$(\delta x, \delta y)$ - independent increments (can be any vector in $\mathbb{R}^2$)

Let $\vec{e} = (e_1, e_2) \in \mathbb{R}^2$ be given. We may restrict ourselves to consider increments which are parallel to $\vec{e}$ only.

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \parallel \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \delta s \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$\delta x = e_1 \delta s, \quad \delta y = e_2 \delta s, \quad \delta s \in \mathbb{R}$$

$$\delta f = f(x_0 + e_1 \delta s, y_0 + e_2 \delta s) - f(x_0, y_0)$$

Can be considered as an increment of the 1-variable function

$$\varphi_{\vec{e}}(s) := f(x_0 + e_1 s, y_0 + e_2 s)$$

$\Rightarrow$ we restrict our function to the line

$$\begin{aligned} x(s) &= x_0 + e_1 s \\ y(s) &= y_0 + e_2 s \end{aligned}$$

parametric equation of the line passing trough (x_0, y_0) and parallel to $\vec{e}$.

$$\delta \varphi_{\vec{e}}(s) = \delta f = (f_x' \cdot e_1 + f_y' \cdot e_2) \delta s + H.O.T.$$

$\Rightarrow$

$$d\varphi_{\vec{e}} = (f_x' \cdot e_1 + f_y' \cdot e_2) ds$$

$$\frac{d\varphi_{\vec{e}}}{ds} = f_x' \cdot e_1 + f_y' \cdot e_2$$

derivative of f in direction $\vec{e}$.

Has a special name: directional derivative and special notation

$$\partial_{\vec{e}} f = \frac{\partial f}{\partial \vec{e}} = \text{partial derivative in the direction of } \vec{e}$$

Examples:

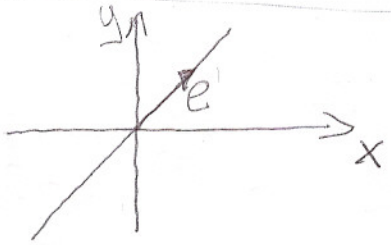
1) $\vec{e} = (1, 0)$ - x-direction

$$\frac{\partial f}{\partial e} = 1 \cdot \frac{\partial f}{\partial x} + 0 \cdot \frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}$$ - partial derivative with respect to x

2) $\vec{e} = (0, 1)$ y-direction

$$\frac{\partial f}{\partial e} = 0 \cdot \frac{\partial f}{\partial x} + 1 \cdot \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}$$ —//— with respect to y.

3) $\vec{e} = (1, 1)$



$$\boxed{\frac{\partial f}{\partial e} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}}$$

Directional derivative $\frac{\partial f}{\partial e}$ measures the growth rate of f in the direction of $\vec{e}$.

We have derived that

$$\boxed{\frac{\partial f}{\partial e} = f'_x \cdot e_1 + f'_y \cdot e_2 = \nabla f \cdot \vec{e}}$$

dot product of 2 vectors

However: if $\vec{e}_1 = \varkappa \vec{e}_2$

$\varkappa \geq 0 \Rightarrow \vec{e}_1$ and $\vec{e}_2$ are parallel and give the same direction, but

$$\boxed{\frac{\partial f}{\partial e_1} = \varkappa \frac{\partial f}{\partial e_2}}$$

Natural to fix the length of vector e

$$\boxed{\|\vec{e}\|^2 = e_1^2 + e_2^2 = 1}$$

Then $s \in \mathbb{R}$ measures the usual distance between points on a line

$$\boxed{\begin{aligned} x(s) &= x_0 + e_1 s \\ y(s) &= y_0 + e_2 s \end{aligned}}$$

indeed

$$\left| \begin{pmatrix} x(s) - x(0) \\ y(s) - y(0) \end{pmatrix} \right|^2 =$$

$$= s^2 |\vec{e}|^2 = s^2$$

In this case, δs is just a distance between (x_0, y_0) and $(x_0 + \delta x, y_0 + \delta y)$

Thus, fixing $\boxed{|\vec{e}| = 1}$

$\frac{\partial f}{\partial e}$ indicates the "reaction" of the function f on increments $(\delta x, \delta y)$ of the same module $\boxed{|(\delta x, \delta y)| = \delta s}$, but in different directions parametrized by the vector $\vec{e}$.

As we see from

$$\boxed{\frac{\partial f}{\partial e} = \nabla f \cdot \vec{e}}$$

A function has different growth rate in different directions.

Natural questions: In what direction the growth rate will be maximal or minimal?

Answer:

from elementary geometry we know that $\boxed{\vec{e} \cdot \vec{m} = |\vec{e}| \cdot |\vec{m}| \cdot \cos \angle(\vec{e}, \vec{m})}$

angle between $\vec{e}$ and $\vec{m}$

Thus, for $|\vec{e}| = 1$, we have

$$\boxed{\frac{\partial f}{\partial e} = |\nabla f| \cdot \cos \angle(\nabla f, \vec{e})}$$

$\Rightarrow$ $\max \frac{\partial f}{\partial e}$ is achieved when $\boxed{\vec{e} \parallel \nabla f}$

Conclusion (geometric meaning of the gradient):

the direction of ∇f gives the direction of the steepest growth of f

the modulus $|\nabla f|$ measures the rate of growth in the steepest growth direction.

Another observation: if $\boxed{\vec{e} \perp \nabla f}$

then $\boxed{\frac{\partial f}{\partial e} = 0}$ $\Rightarrow$ growth rate in this direction is $O(\delta s)$ (from H.O.T)

$\Rightarrow \vec{e}$ is the direction of "slowest" growth rate.

This can be visualized using contour plots and contour lines

Instead of ploting the 3D graph of

$$\boxed{z = f(x, y)}$$

One may plot the contour lines

$$\boxed{f(x, y) = C}$$

on the 2D plane (x, y) for different values of C (for instance, using different colours)

a line which corresponds to all $(x, y) \in \mathbb{R}$ which give the same value of z

Th. $\nabla f \perp$ tangent line to $\boxed{f(x, y) = C}$

Why? Let $f(x_0, y_0) = C$ and let the increment $(\delta x, \delta y)$ is such that $f(x_0 + \delta x, y_0 + \delta y) = C$ (belongs to the same contour line)

Then

$$\boxed{0 = \delta f = f'_x \delta x + f'_y \delta y + H.O.T}$$ $\rightarrow$ infinitesimal increments

$$\boxed{f'_x dx + f'_y dy = 0}$$ equation for tangent line $=$ $\boxed{f'_x \cdot (x - x_0) + f'_y \cdot (y - y_0) = 0}$

the same, but in more usual form

$$\| $$

$$\nabla f \cdot (dx, dy) = 0$$

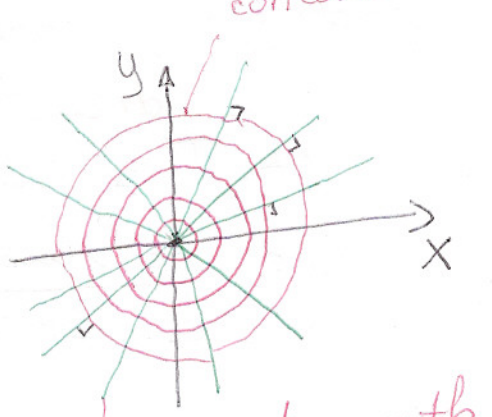
the tangent line direction

Example:

$\boxed{z = x^2 + y^2}$ (circular paraboloid)

$\boxed{x^2 + y^2 = C}$ - concentric circles

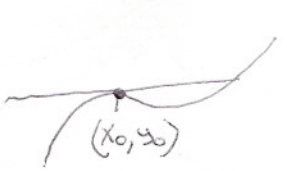
$\boxed{\nabla f = (2x, 2y)}$ - gives the direction of a line passing through 0 and (x, y)



In last examples, the steepest growth curves were straight lines $y = kx$, $k \in \mathbb{R}$. That is not true in general!

Indeed,

Steepest growth curve = **tangent line at every point is parallel to the gradient of f at that point**



tangent line is generated by the vector

$$(dx, dy) = (1, y')dx$$

now y is dependent var

so the condition is

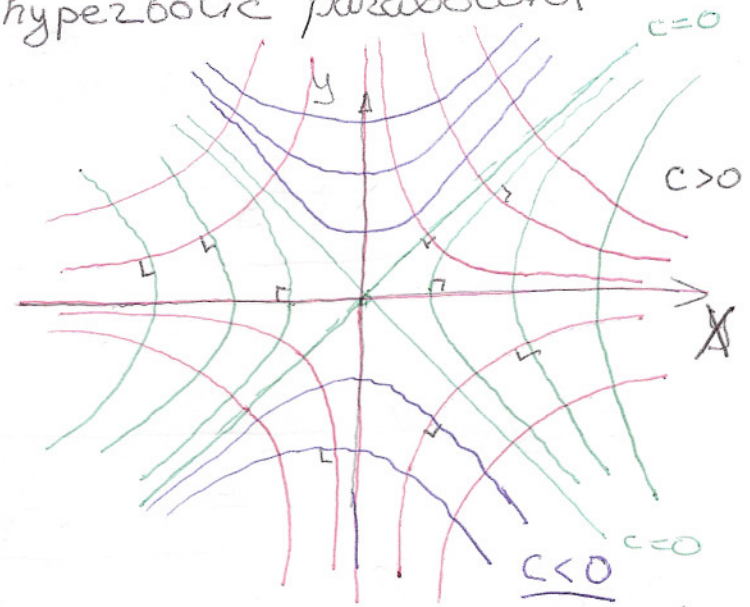
$$\begin{pmatrix} dx \\ dy \end{pmatrix} \parallel \begin{pmatrix} f'_x \\ f'_y \end{pmatrix} \quad \text{or} \quad \begin{aligned} dx &= \varkappa f'_x \\ dy &= \varkappa f'_y \end{aligned}$$

or

$$\frac{dy}{dx} = \frac{f'_y}{f'_x}$$ – Differential equation for the steepest growth curves

$\Rightarrow$ **Steepest growth curves** = **Integral curves of that ODE**

Example: $z = x^2 - y^2$ – hyperbolic paraboloid



green, Blue = contour curves
red = steepest descent curves.

contour curves:

$$x^2 - y^2 = C$$

= hyperbolas with oblique asymptotes

$y = \pm x$ (corresponds to $c=0$)

– check!

Steepest descent curves:

$\nabla f = (2x, -2y)$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$x\,dy + y\,dx = 0$

$d(xy) = 0$

$$xy = C$$ – family of hyperbolas with oblique asymptotes $x=0$ and $y=0$

Example: $z = 2x^2 + y^2$

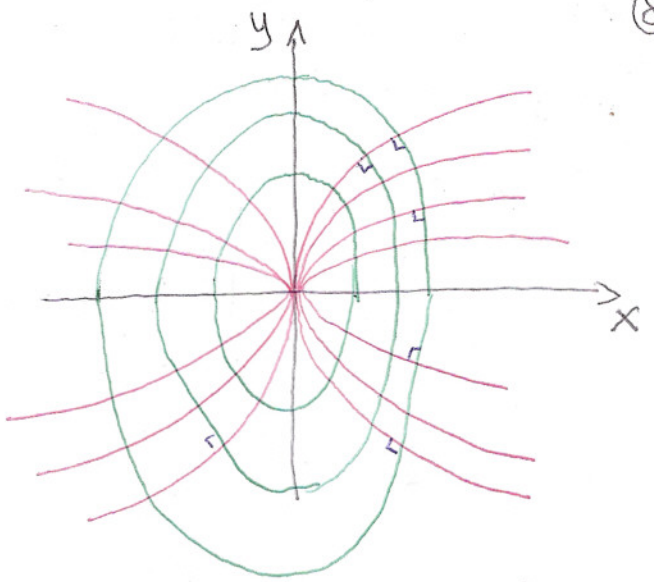
elliptic paraboloid

contour lines:

$2x^2 + y^2 = C$

family of ellipses

$\nabla z = (\partial/x, 2y)$



green = contour curves

red = steepest growth curves

$\frac{dy}{dx} = \frac{y}{2x}$ $\quad 2x\,dy - y\,dx = 0$

$\frac{dy}{y} = \frac{dx}{2x}$ $\quad \log y = \frac{1}{2}\log x + C$

$\frac{x}{y^2} = C$ - family of parabolas

Extension: functions of more than two variables.

say 3 variables.

$f = f(x, y, z)$

Now, contour lines are replaced by contour surfaces in 3D

$(x, y, z) \in \mathbb{R}^3$

$f(x, y, z) = C \in \mathbb{R}$

Equation for the contour surfaces.

Tangent plane: at (x_0, y_0, z_0)

$0 = f'_x\,dx + f'_y\,dy + f'_z\,dz$ OR $0 = f'_x(x - x_0) + f'_y(y - y_0) + f'_z(z - z_0)$

All derivatives are computed at (x_0, y_0, z_0)!

Important formula!

Gradient $\nabla f = (f'_x, f'_y, f'_y)$

Ortogonal to the tangent plane

as in the 2D case.

Reminder: vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is orthogonal to the plane $ax + by + cz + d = 0$ $\Rightarrow$ Linear algebra!