Lecture 10-11. Chain Rule and change of variables in differential operators

Aim: - to extend the chain rule to many variables
- to learn how to express partial derivatives with respect to "old" variables in terms of "new" variables.

Reminder: the case of 1-variable:

$\boxed{y = y(x)}$ x - is independent variable
y - is dependent variable

Want to change independent variable $\boxed{x = x(t)}$

Now

t (independent) $\longrightarrow$ x (intermediate) $\longrightarrow$ y (dependent)

Consider $t = t_0 \to x_0 = x(t_0) \to y_0 = y(x_0)$ and look at our maps near that points. Let $\boxed{\delta t = t - t_0}$ - small increment of t (independent)

$\boxed{\delta x = x - x_0}$ - increment of x - intermediate $\boxed{\delta y = y - y_0}$ increment of y (dependent)

We know that

(**) $\boxed{\delta x = x'(t_0)\delta t + H.O.T.}$ $\boxed{H.O.T = O(\delta t^2)}$ higher order term in variable t!

(*) $\boxed{\delta y = y'(x_0)\delta x + H.O.T.}$ $\leftarrow$ $\boxed{H.O.T = O(\delta x^2)}$ - Higher order terms with respect to x

But $\boxed{O(\delta x^2) = O(\delta t^2)}$ (δx is of order δt according to (**))

Inserting (**) to (*)

$$\boxed{\delta y = y'(x_0)[x'(t_0)\delta t + H.O.T.] + H.O.T = y'(x_0)x'(t_0)\delta t + H.O.T.}$$

$$\Rightarrow \quad \boxed{dy = y'(x_0)x'(t_0)dt} \qquad \boxed{\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}}$$

$\Rightarrow$ Usual scalar chain rule! with "almost" rigorous proof. One more interpretation:

Invariance of the first differential: it does not matter in

(D) $\boxed{dy = y'(x)\,dx}$

whether x is independent, dependent or intermediate

Indeed, if you want to make x independent and y - independent, you just write

$$dx = [y'(x_0)]^{-1} dy \qquad x'(y_0) = [y'(x_0)]^{-1}$$

inverse function

And if you make x intermediate you just insert $dx = x'(t)dt$ to (D)

$$dy = y'(x)\,x'(t)dt$$ - chain rule

In ALL that cases (D) is valid.

For comparison, the second differential is not invariant:

(D^2) $\quad d^2y = y''(x)\,dx^2$ $\quad x$ is independent

But if we make x intermediate: $dx = x'(t)dt$ and insert this to (D^2), we have

$$d^2y = y''(x)[x'(t)]^2 dt^2$$ — WRONG!

since

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(y'(x)\,x'(t)\right) = y''(x)[x'(t)]^2 + y'(x)x''(t)$$

extra term

$\Rightarrow$ We cannot use (D^2) if x is not an independent variable! Extra term arises!

Where this extra term comes from?

Expand:

$$\delta x = x'(t)\delta t + \frac{1}{2}x''(t)\delta t^2 + O(\delta t^3)$$

and

$$\delta y = y'(x)\delta x + \frac{1}{2}y''(x)\delta x^2 + O(\delta x^3)$$

Inserting δx into the formula for δy

$$\delta y = y'(x)\left[x'(t)\delta t + \frac{1}{2}x''(t)\delta t^2\right] + \frac{1}{2}y''(x)\left[x'(t)\delta t\right]^2 + \text{H.O.T.} =$$

$$= \underbrace{y'(x)x'(t)\delta t}_{\text{1-st order}} + \underbrace{\frac{1}{2}\left[y'(x)x''(t) + y''(x)[x'(t)]^2\right]\delta t^2}_{\text{2nd order}} + \dots$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} = y'(x)\,x''(t) + y''(x)[x'(t)]^2}$$

the extra term comes from the 2-nd order correction of the Linear term.

Return to the general case:

$y = y(x)$, but now $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

Want to change: $x = x(t)$, $t \in \mathbb{R}^k$

$t \in \mathbb{R}^k$ (independent) $\longrightarrow$ $x \in \mathbb{R}^n$ (intermediate) $\longrightarrow$ $y \in \mathbb{R}^m$

Everything is very similar to the scalar case

Consider again a point $t_0 \in \mathbb{R}^k \to x_0 \in \mathbb{R}^n \to y_0 \in \mathbb{R}^m$ and small increments near these points

$\delta t = t - t_0$ - k-vector (increment of independent variable)

$\delta x = x - x_0$ - n-vector (increment of intermediate variable)

$\delta y = y - y_0$ - m-vector (increment of dependent variable)

We know that

$$\delta x = J_t(x)\delta t + H.O.T.$$

$$J_t(x) = \frac{\partial(x_1, \dots x_n)}{\partial(t_1, \dots t_k)} = \begin{pmatrix} x'_{1,t_1}, \dots x'_{1,t_k} \\ \vdots \\ x'_{n,t_1}, \dots x'_{n,t_k} \end{pmatrix}$$

$n \times k$ - matrix

Jacobi matrix of the map $x = x(t)$

and that

$$\delta y = J_x(y)\delta x + H.O.T$$

$$J_x(y) = \frac{\partial(y_1, \dots y_m)}{\partial(x_1, \dots x_n)} = \begin{pmatrix} y'_{1,x_1}, \dots, y'_{1,x_n} \\ \vdots \\ y'_{m,x_1}, \dots, y'_{m,x_n} \end{pmatrix}$$

Jacobi matrix of the map $y = y(x)$

Inserting δx into the formula for δy, we see

$$\delta y = J_x(y)\left[J_t(x)\delta x + \dots\right] + H.O.T. = J_x(y) \cdot J_t(x)\delta t + H.O.T.$$

matrix multiplication!

neglecting H.O.T. $\Rightarrow$

$$dy = J_x(y) \cdot J_t(x)dt \Rightarrow J_t(y) = J_x(y) \cdot J_t(x)$$

$n \times k \quad n \times n \quad n \times k$

correct for matrix multiplication

$J_t(y)$ - $m \times k$ matrix - Jacobi matrix of a compound map $y = y(x(t))$

The desired extension of a chain rule for many variables

OR

$$\frac{\partial(y_1, \dots, y_m)}{\partial(t_1, \dots t_k)} = \frac{\partial(y_1, \dots y_m)}{\partial(x_1, \dots, x_n)} \cdot \frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots t_k)}$$

most natural - in the matrix form!!!

We see the matrix analogue of $\boxed{\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}}$

$$\boxed{\frac{\partial(y_1,\dots,y_m)}{\partial(t_1,\dots,t_k)} = \frac{\partial(y_1,\dots,y_m)}{\partial(x_1,\dots,x_n)}\cdot\frac{\partial(x_1,\dots,x_n)}{\partial(t_1,\dots,t_k)}}$$

NB! matricies do not commute! the order of the multipliers is important.

This chain rule formula allows to compute ALL partial derivatives for the compound function

$$\boxed{y = y(x(t))}$$

In applications often it is simpler to use the invariance of the 1st differential:

(D) $$\boxed{dy = y'_x(y)\,dx}$$

Again, $\boxed{dy}$ and $\boxed{dx}$ can be independent, dependent or intermediate (D) is true in all cases. We just need to insert the proper expression for $\boxed{dx}$ (or $\boxed{dy}$) in it exactly as in 1D case.

Examples:

1) $\boxed{f: \mathbb{R}^2 \to \mathbb{R}}$ - scalar function of 2 variables

$\boxed{f = f(x,y)}$ (different x, y than in general theory, just two scalar variables!)

+ change of variables

$$\boxed{x = x(u,v),\quad y = y(u,v)}$$

Compound function:

$\boxed{z = f(x(u,v), y(u,v))}$ want to find partial derivatives with respect to u and v.

Solution: use the invariance of the 1-st differential

$$dz = z'_u\,du + z'_v\,dv = f'_x\,dx + f'_y\,dy = f'_x\left[x'_u\,du + x'_v\,dv\right] +$$

$$+ f'_y\left[y'_u\,du + y'_v\,dv\right] =$$

$$= \left[f'_x\cdot x'_u + f'_y\cdot y'_u\right]du + \left[f'_x\cdot x'_v + f'_y\cdot y'_v\right]dv$$

$\boxed{dx = x'_u\,du + x'_v\,dv}$
$\boxed{dy = y'_u\,du + y'_v\,dv}$

Since $\boxed{du}$ and $\boxed{dv}$ are independent

$$f'_u = \boxed{z'_u = f'_x\cdot x'_u + f'_y\cdot y'_u}$$

$$f'_v = \boxed{z'_v = f'_x\cdot x'_v + f'_y\cdot y'_v}$$

OR

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

in vector notations

$$\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$

1 x 2 matrix; 1 x 2 matrix; 2 x 2 - matrix

in a complete agreement with general theory !

2) More concrete example - polar coordinates

$f = f(x,y)$, $x = R\cos\varphi$, $y = R\sin\varphi$

compound function

$$f(R,\varphi) = f(R\cos\varphi, R\sin\varphi)$$

$$\frac{\partial f}{\partial R} = \frac{\partial f}{\partial x}\cdot\cos\varphi + \frac{\partial f}{\partial y}\cdot\sin\varphi$$
$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x}\cdot R(-\sin\varphi) + \frac{\partial f}{\partial y}\cdot R\cos\varphi$$

$\frac{\partial x}{\partial R} = \cos\varphi$, $\frac{\partial x}{\partial \varphi} = -R\sin\varphi$

$\frac{\partial y}{\partial R} = \sin\varphi$, $\frac{\partial y}{\partial \varphi} = R\cos\varphi$

3) Linear change of variables

$$x = u+v$$
$$y = u-v$$

$f(u,v) = f(u+v, u-v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

4) Let $f = f(x,y,z)$ $f: \mathbb{R}^3 \to \mathbb{R}$ and

$x = x(t)$, $y = y(t)$, $z = z(t)$ $t \in \mathbb{R}$ - scalar variable.

Compound function

$$f(t) = f(x(t), y(t), z(t))$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot x'(t) + \frac{\partial f}{\partial y}\cdot y'(t) + \frac{\partial f}{\partial z}\cdot z'(t)$$

5) Derivatives of by-linear functions of vector arguments.

a) Dot' product of two vectors:

$f(u,v) = u.v$, $u = \begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$, $v = \begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$ $u.v = u_1v_1 + \dots + u_nv_n$

By Leibnitz Rule:

$$d(u.v) = d(u_1v_1 + \cdots + u_nv_n) = d(u_1v_1) + \cdots + d(u_nv_n) =$$

$$= u_1\,dv_1 + v_1\,du_1 + \cdots + u_n\,dv_n + v_n\,du_n = du.v + u.dv$$

$$\boxed{d(u.v) = du.v + u.dv}$$ Leibnitz Rule for the dot-product

Now, let vectors $u(t), v(t) \in \mathbb{R}^n$ depend on time t - scalar variable. Then using the invariance of 1st diff

$$\boxed{\frac{d}{dt}\left(u(t).v(t)\right) = \frac{du}{dt}\cdot v(t) + u(t).\frac{dv}{dt}}$$

In particular

$$\boxed{\frac{d}{dt}|u(t)|^2 = \frac{d}{dt}\left(u(t).u(t)\right) = 2\,u(t).\frac{du}{dt}}$$

B) Cross-product: $u \times v$ - vector function of 2-vectors $u, v \in \mathbb{R}^3$

$$\boxed{u \times v = \begin{vmatrix} e_1, e_2, e_3 \\ u_1, u_2, u_3 \\ v_1, v_2, v_3 \end{vmatrix}}$$

Again, linear with respect to u and v

By Leibnitz rule

$$\boxed{d(u \times v) = du \times v + u \times dv}$$ - check!

If vectors $u(t), v(t)$ depend on time t

$$\boxed{\frac{d}{dt}\left(u(t) \times v(t)\right) = \frac{du}{dt} \times v(t) + u(t) \times \frac{dv}{dt}}$$

Change of variables in differential operators:

Problem: An unknown function $f(x_1, \ldots, x_n)$ satisfies some equation containing partial derivatives (= partial differential equation). For instance, in 2-variables

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$ - transport equation

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ - Laplace equation

$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}$ - wave equation

And write the equations in new variables

Want to change independent variable here

$$\boxed{\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned}}$$ For example, polar coordinates

OR something more general

Need to know: how to express partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the unknown f through its partial derivatives $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$

Solution: we have a function $f = f(x, y)$ of 2-variables and a change of variables

$$\boxed{\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \end{aligned}}$$

The function f in variables (u, v) reads

$$\boxed{f(u,v) = f(x(u,v), y(u,v))} \text{ – compound function}$$

By the Chain Rule

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}}$$

Almost done!

But now $\frac{\partial}{\partial u} f$ and $\frac{\partial}{\partial v} f$ are expressed in terms of $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$ $\Rightarrow$ need to solve the Linear system with respect to $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial x} f$

$$\begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

↑ Transposed Jacobi matrix

$$\Rightarrow \boxed{\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \left(\left[\frac{\partial(x,y)}{\partial(u,v)} \right]^t \right)^{-1} \begin{pmatrix} \partial f/\partial u \\ \partial f/\partial v \end{pmatrix}}$$

OR omitting f, we end up with general formula for diff operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

$$\boxed{\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \left(\left[\frac{\partial(x,y)}{\partial(u,v)} \right]^t \right)^{-1} \begin{pmatrix} \partial/\partial u \\ \partial/\partial v \end{pmatrix}}$$

general formula. Generalization to any dimension is immediate.

Examples:

1) Polar coordinates:

Need to solve this system:

$$\begin{cases} \frac{\partial f}{\partial R} = \cos\varphi \cdot \frac{\partial f}{\partial x} + \sin\varphi \cdot \frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial \varphi} = -R\sin\varphi \cdot \frac{\partial f}{\partial x} + R\cos\varphi \cdot \frac{\partial f}{\partial y} \end{cases}$$

Multiplying 1st equation on $R\cos\varphi$, the second on $-\sin\varphi$ and taking a sum

$$\Rightarrow R\frac{\partial f}{\partial x} = R\cos\varphi \frac{\partial f}{\partial R} - \sin\varphi \frac{\partial f}{\partial \varphi}$$

$$\frac{\partial f}{\partial x} = \cos\varphi \cdot \frac{\partial f}{\partial R} - \frac{1}{R}\sin\varphi \cdot \frac{\partial f}{\partial \varphi}$$

Analogously: $\frac{\partial f}{\partial y} = \sin\varphi \cdot \frac{\partial f}{\partial R} + \frac{1}{R}\cos\varphi \cdot \frac{\partial f}{\partial \varphi}$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\varphi \cdot \frac{\partial}{\partial R} - \frac{1}{R}\sin\varphi \cdot \frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial y} &= \sin\varphi \cdot \frac{\partial}{\partial R} + \frac{1}{R}\cos\varphi \cdot \frac{\partial}{\partial \varphi} \end{aligned}}$$

Laplace operator in polar coordinates: not necessary for the exam!

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f\right) = \left(\cos\varphi \cdot \frac{\partial}{\partial R} - \frac{1}{R}\sin\varphi \cdot \frac{\partial}{\partial \varphi}\right)($$

$$\left(\cos\varphi \cdot \frac{\partial f}{\partial R} - \frac{1}{R}\sin\varphi \cdot \frac{\partial}{\partial \varphi} f\right) + \left(\sin\varphi \cdot \frac{\partial}{\partial R} + \frac{1}{R}\cos\varphi \frac{\partial}{\partial \varphi}\right)\left(\sin\varphi \cdot \frac{\partial f}{\partial R} + \frac{1}{R}\cos\varphi \frac{\partial f}{\partial \varphi}\right)$$

$$= \cos\varphi \cdot \frac{\partial}{\partial R}\left(\cos\varphi \frac{\partial f}{\partial R} - \frac{1}{R}\sin\varphi \cdot \frac{\partial f}{\partial \varphi}\right) - \frac{1}{R}\sin\varphi \cdot \frac{\partial}{\partial \varphi}\left(\cos\varphi \frac{\partial f}{\partial R} - \frac{1}{R}\sin\varphi \frac{\partial f}{\partial \varphi}\right) +$$

$$+ \sin\varphi \frac{\partial}{\partial R}\left(\sin\varphi \cdot \frac{\partial f}{\partial R} + \frac{1}{R}\cos\varphi \cdot \frac{\partial f}{\partial \varphi}\right) + \frac{1}{R}\cos\varphi \frac{\partial}{\partial \varphi}\left(\sin\varphi \cdot \frac{\partial f}{\partial R} + \frac{1}{R}\cos\varphi \cdot \frac{\partial f}{\partial \varphi}\right) =$$

$$= \cos^2\varphi \frac{\partial^2 f}{\partial R^2} + \frac{1}{R^2}\cos\varphi \sin\varphi \frac{\partial f}{\partial \varphi} - \frac{1}{R}\cos\varphi \sin\varphi \frac{\partial^2 f}{\partial R \partial \varphi} + \frac{1}{R}\sin^2\varphi \frac{\partial f}{\partial R} -$$

$$- \frac{1}{R}\sin\varphi \cos\varphi \frac{\partial^2 f}{\partial R \partial \varphi} + \frac{1}{R^2}\sin\varphi \cos\varphi \frac{\partial f}{\partial \varphi} + \frac{1}{R^2}\sin^2\varphi \frac{\partial^2 f}{\partial \varphi^2} +$$

$$+ \sin^2\varphi \frac{\partial^2 f}{\partial R^2} - \frac{1}{R^2}\cos\varphi \sin\varphi \frac{\partial f}{\partial \varphi} + \frac{1}{R}\sin\varphi \cos\varphi \frac{\partial^2 f}{\partial \varphi \partial R} + \frac{1}{R}\cos^2\varphi \frac{\partial f}{\partial R} + \frac{1}{R} \times$$

$$\times \cos\varphi \sin\varphi \frac{\partial^2 f}{\partial R \partial \varphi} - \frac{1}{R^2}\cos\varphi \sin\varphi \frac{\partial f}{\partial \varphi} + \frac{1}{R^2}\cos^2\varphi \frac{\partial^2 f}{\partial \varphi^2} =$$

$$= \frac{\partial^2 f}{\partial R^2} + \frac{1}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \varphi^2}$$ - terrible computations, but simple answer

$$\boxed{\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial R^2} + \frac{1}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \varphi^2}}$$

Very useful formula! In particular, allows to find radially symmetric solutions $f = f(R)$.

Example 2) (the simple one)

$$\begin{cases} x = u+v \\ y = u-v \end{cases}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left[\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right], \quad \frac{\partial f}{\partial y} = \frac{1}{2}\left[\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}\right]$$

$$\frac{\partial}{\partial x} = \frac{1}{2}\left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right], \quad \frac{\partial}{\partial y} = \frac{1}{2}\left[\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right]$$

a) Let us find the expression for the wave operator (d'Alembert operator)

$$\Box f := \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f\right) = \frac{1}{4}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) -$$

$$- \frac{1}{4}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} f - \frac{\partial}{\partial v} f\right) = \frac{1}{4}\left[\frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right] - \frac{1}{4}\left[\frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}\right] = \frac{\partial^2 f}{\partial u \cdot \partial v} \Rightarrow$$

$$\Box f = \frac{\partial^2 f}{\partial u \cdot \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)$$

In that coordinates the wave equation $\Box f = 0$ can be solved by subsequent integration with respect to u and v!

b) Transport equation:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0, \text{ but } \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \Rightarrow$$

$$\Rightarrow \frac{\partial f}{\partial u} = 0 \text{ integrating in } v \quad f(u,v) = \varphi(v)$$

(φ: arbitrary function)

Returning back to variables (x, y)

$$v = \frac{x-y}{2}$$

we obtain a general solution

$$f(x,y) = \varphi(x-y)$$ - depends on arbitrary function of 1 variable.