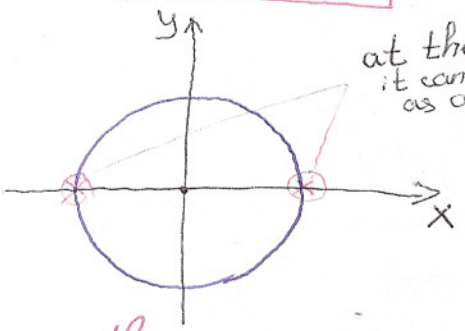
Lecture 12-13 Implicit functions and implicit function theorems

Two examples:

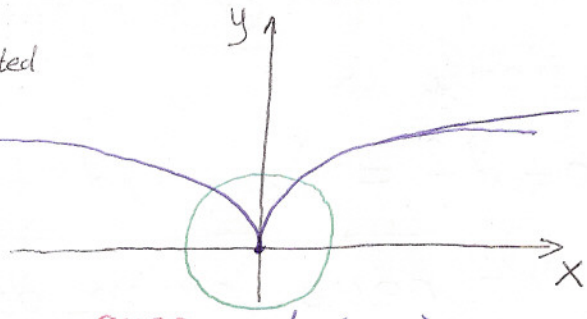
I $\boxed{x^2+y^2=1}$



at these points it cannot be presented as a graph $y=y(x)$!

smooth curve (just a circle)

II $\boxed{x^2-y^3=0}$ $y=\sqrt[3]{x^2}$



cusp at $(0,0)$

singular points on the curve

Questions: 1) Why the singular points may appear on a curve given by the smooth function f?

$$\boxed{f(x,y)=0}$$

2) How to find them?

OR need to understand when the curve $\boxed{f(x,y)=0}$ is smooth near (x_0,y_0), $\boxed{f(x_0,y_0)=0}$

To this end, we want to present the curve locally as a graph of a smooth function

$$\boxed{\begin{matrix} y=y(x) \\ y_0=y(x_0)\end{matrix}} \quad \boxed{f(x,y(x))\equiv 0} \quad \text{OR} \quad \boxed{\begin{matrix} x=x(y) \\ x_0=x(y_0)\end{matrix}} \quad \boxed{f(x(y),y)\equiv 0}$$

Warning: Usually, we can find such functions only locally! Since not ALL curves are graphs of functions!

Look at a circle!

$\boxed{x^2+y^2=1}$ $\boxed{y(x)=\pm\sqrt{1-x^2}}$ $\Rightarrow$ two functions

$\boxed{y_+(x)=\sqrt{1-x^2}}$ and $\boxed{y_-(x)=-\sqrt{1-x^2}}$

When $\boxed{x=\pm 1}$ we cannot write a circle as $y=y(x)$ with a smooth (single valued) function $y(x)$.

But, near that points we may write it as $x=x(y)$

$\boxed{x=-\sqrt{1-y^2}}$ near $(-1,0)$ And $\boxed{x=+\sqrt{1-y^2}}$ near $(1,0)$

$\Rightarrow$ indeed a smooth curve at every point.

General (scalar) case

$f(x,y)=0$ near (x_0,y_0)

Want to find smooth $y=y(x)$, $y(x_0)=y_0$ such that

$f(x,y(x))\equiv 0$ near $x=x_0$

Local theory $\Rightarrow$ may start with linearization

$$0=df=f_x'(x_0,y_0)\,dx+f_y'(x_0,y_0)\,dy$$

$\Rightarrow$! We need extra assumption

(A) $f_y'(x_0,y_0)\neq 0$

In order to be able to express dy through dx

$$dy=-\frac{f_x'(x_0,y_0)}{f_y'(x_0,y_0)}\,dx$$

OR

$$y'(x_0)=-\frac{f_x'(x_0,y_0)}{f_y'(x_0,y_0)}$$

if the desired $y(x)$ exists.

In addition, if (A) is satisfied

$\Rightarrow$ $f_y'(x,y)\neq 0$ for all (x,y) close to (x_0,y_0)

Diff $f(x,y(x))\equiv 0 \Rightarrow f_x'+f_y'\,y'(x)=0$

OR

$$y'(x)=-\frac{f_x'(x,y(x))}{f_y'(x,y(x))}$$

for x close to x_0 (differential equation for $y(x)$!)

Question: Is (A) sufficient for the existence of a smooth $y=y(x)$?

Answer: Yes! is given by

Implicit function theorem (scalar case):

Let $f=f(x,y)$ be smooth and let $(x_0,y_0)\in\mathbb{R}^2$ be such that $f(x_0,y_0)=0$, $f_y'(x_0,y_0)\neq 0$

$\Rightarrow$ there exists a uniquelly defined and smooth function $y=y(x)$ such that

$y_0=y(x_0)$ and $f(x,y(x))\equiv 0$ for all x close to x_0.

Deep Th. from Real Analysis!!!

Some heuristics or how to construct the desired $y = y(x)$ (the modern proof is different and uses the so-called Banach contraction (will be studied at Year 3!))

Let us try to find the Taylor expansion of $y(x)$

$$y(x) = y_0 + y'(x_0)(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \dots$$

$\Rightarrow$ We only need to compute all derivatives $y^{(n)}(x_0)$ and define

$$y(x) := \sum_{n=0}^{\infty}{}' \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$$

$y'(x_0) = -\frac{f'_x}{f'_y}\Big|_{(x_0,y_0)}$ is already found.

$n=2$

$$y''(x) = \frac{d}{dx}\left(-\frac{f'_x(x,y(x))}{f'_y(x,y(x))}\right) = -\frac{\frac{d}{dx}f'_x(x,y(x))\cdot f'_y - \frac{d}{dx}f'_y \cdot f'_x}{(f'_y)^2} =$$

$$= \frac{\left[f''_{xx} + f''_{xy}\cdot y'(x)\right]f'_y - \left[f''_{xy} + f''_{yy}\cdot y'(x)\right]f'_x}{(f'_y)^2}$$

$\Rightarrow$ Only f'_y (in some power) in the denominator

$\Rightarrow$ if $f'_y(x_0,y_0) \neq 0$ — we may find $y''(x_0)$

$$y''(x_0) = -\frac{(f'_y\cdot f''_{xx} - f'_{xy}\cdot f'_x)f'_y - (f'_y\cdot f'_{xy} - f'_{yy}\cdot f'_x)f'_x}{(f'_y)^3}\Bigg|_{(x_0,y_0)}$$

more and more complicated expressions for higher derivatives, $y'''(x_0)$, $y^{(4)}(x_0)$, ... But ALL of them can be computed !!!

$\Rightarrow$ ALL coefficients in Taylor expansion of $y(x)$ can be found $\Rightarrow$ $y(x)$ can be found

Not a rigorous proof! Why? Taylor series may have ZERO radius of convergence! (see Real Analysis)

$\Rightarrow$ need to prove that the radius of convergence is not ZERO! This can be done if f is analytic!

Conclusions:

$f(x,y)=0$ near (x_0,y_0)

if $f_y'(x_0,y_0)\neq 0$ $\Rightarrow$ the curve is smooth near (x_0,y_0) - graph $y=y(x)$

What happens if $f_y'(x_0,y_0)=0$?

We may try $x=x(y)$!

Implicit function theorem $\Rightarrow$

if $f_x'(x_0,y_0)\neq 0$ the curve is smooth near (x_0,y_0) - graph $x=x(y)$

$x=x(y)$ exists and smooth

$\Rightarrow$ NB!! (x_0,y_0) ($f(x_0,y_0)=0$) can be a singular point of the curve $f(x,y)=0$

only if

$\nabla f(x_0,y_0)=0$

equation (2 equations) on singular points on the curve $f(x,y)=0$

The 3rd equation is, of course, $f(x_0,y_0)=0$

Examples:

(I) $x^2+y^2=1$

$\nabla f=(2x,2y)$

$\nabla f=0 \Rightarrow (x,y)=(0,0)$

this point is not on the curve $\Rightarrow$ no singular points $\Rightarrow$ smooth curve!

(II) $x^2-y^3=0$

$\nabla f=(2x,-3y^2)$

$\nabla f=0 \Rightarrow (x,y)=(0,0)$

this point IS on the curve!

$\Rightarrow$ (0,0) is indeed a singular point (cusp).

(III) $x^3-y^3=0 \Leftrightarrow x^3=y^3 \Leftrightarrow y=x$ - smooth curve (straight line)

However, $\nabla f=(3x^2,-3y^2)$ $\nabla f|_{(0,0)}=0$ $\Rightarrow$ despite this - no singularities at (0,0)!

Thus, points where $\nabla f=0$ can be regular, but singularities may appear only at such points! Other points (where $\nabla f\neq 0$) are regular.

$\Rightarrow$ more detailed analysis is required for every point of $\nabla f=0$

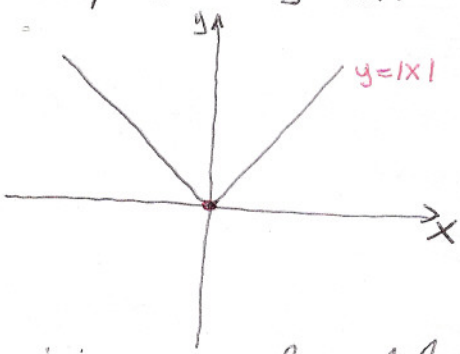
Reminder: Potential singular points should be included to the list of critical points!! (NB!!!)

Why? Maxima and minima "like" to be exactly at such singular points!!

Forgetting this leads to mistakes! (including the exam).

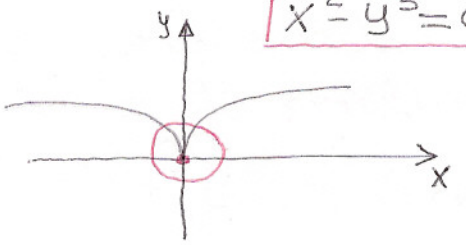
Examples:

"Simple" $y=|x|$



minimum (the global one) is achived at $x=0$ where the function is not smooth $\Rightarrow$ will be lost at the 1st order test!!! (do not satisfy $f'(x)=0$)

"more advanced": minimize $f(y)=y$ under the constrain $x^2-y^3=0$



the global minimum "lives" at the cusp point $(0,0)$. Cannot be found as an extrema of the Lagrange function (check!) and should be found as a singular point of the constrain curve ($\nabla g=0$). Do not forget this!!!

Multi-dimensional case (Briefly)

We have alreddy discuss the inverse functions and the inverse function theorem:

Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be smooth and $x_0\in\mathbb{R}^n$ be such that $\det J(f)\neq 0$ at $x=x_0$ $\Rightarrow$ the inverse function

$f^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ exists locally near $y_0=f(x_0)$

(should be small neighbourhood of y_0, being rigorous!)

$\Rightarrow$ f is locally one-to-one near $x=x_0$.

Particular case of a general implicit function theorem!

Indeed, let $y=f(x)$. To find the inverse $x=f^{-1}(y)$ we need to solve $y-f(x)\equiv 0$ w.r.t. x $\Rightarrow$ $F(x,y):=y-f(x)$ $\Rightarrow$ need to find $x=x(y)$ such that $F(x(y),y)\equiv 0$ $\Rightarrow$ implicit function!

General multi-dimensional case: (not for the exam!)

Consider a function $f = f(x, y)$ of two vector arguments $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$ and vector-valued: $f \in \mathbb{R}^n$. Let $x_0 \in \mathbb{R}^k$, $y_0 \in \mathbb{R}^n$ be such that

$$f(x_0, y_0) = 0$$

Want to solve the equation $f(x, y) = 0$ near (x_0, y_0) with respect to y = to find a smooth function $y = y(x)$ such that $f(x, y(x)) \equiv 0$.

NB! Notations are the same as before, but now it is a system of n-simultanious equations

$$(*) \quad \begin{cases} f_1(x_1, \dots x_k, y_1, \dots y_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_k, y_1, \dots, y_n) = 0 \end{cases}$$

to be solved with respect to the unknowns $y_1, \dots y_n$.
A non-linear system in contrast to Linear Algebra!

Again, we need to look at the linearization near (x_0, y_0)
To simplify the notations, introduce 2 Jacobians

$$J_x(f) = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_k)}$$ - $n \times k$ - matrix of partial derivatives w.r.t. x

$$J_y(f) = \frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots, y_n)}$$ - $n \times n$ - matrix of partial derivatives w.r.t y.

Then, the total differential of f reads

$$0 = df = J_x(f)\, dx + J_y(f)\, dy \qquad dx = \begin{pmatrix} dx_1 \\ \vdots \\ dx_k \end{pmatrix}, \; dy = \begin{pmatrix} dy_1 \\ \vdots \\ dy_n \end{pmatrix}$$

This equation can be solved with respect to dy if $J_y(f)\big|_{(x_0, y_0)}$ is invertable $= \det \frac{\partial(f_1, \dots f_n)}{\partial(y_1, \dots y_n)}\Big|_{(x_0, y_0)} \neq 0$

In that case

$$dy = [J_y(f)]^{-1} \cdot J_x(f)\, dx$$

OR for the desired function $y = y(x)$

$$J_x(y) := \frac{\partial(y_1, \dots y_n)}{\partial(x_1, \dots x_k)} = \left[\frac{\partial(f_1, \dots, f_n)}{\partial(y_1, \dots y_n)}\right]^{-1} \cdot \frac{\partial(f_1, \dots f_n)}{\partial(x_1, \dots x_k)}$$

Allows to find ALL partial derivatives

$$\frac{\partial y_i}{\partial x_j}\Big|_{x = x_0}$$ for the desired implicit function $y = y(x)$.

Higher partial derivatives can be also found as in the scalar case. ⇒ We can expect that $y=y(x)$ exists and smooth. It is indeed true

Implicit function theorem (general case)

Let $f=f(x,y)$ be an n-vector function of 2 vector variables $x\in\mathbb{R}^k$ and $y\in\mathbb{R}^n$. Let also $x_0\in\mathbb{R}^k$ and $y_0\in\mathbb{R}^n$ be such that $f(x_0,y_0)=0$ and

$$\det \frac{\partial(f_1,\dots,f_n)}{\partial(y_1,\dots,y_n)}\Big|_{(x_0,y_0)} \neq 0$$

⇒ there exists a unique smooth function $y=y(x)$ $(\mathbb{R}^k\to\mathbb{R}^n)$ such that

$$y_0=y(x_0)$$
$$f(x,y(x))\equiv 0$$

and its Jacobian can be computed via the chain Rule

$$\frac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots x_k)} = \left[\frac{\partial(f_1,\dots,f_n)}{\partial(y_1,\dots,y_n)}\right]^{-1}\cdot\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_k)}$$

Again, the formula for derivatives is just a chain Rule, but the existence for $y=y(x)$ is a deep fact from RA.

Extremely useful theorem with huge amount of applications (not for that module!)

Here, we do only one example:

Let the functions $x=x(u,v)$ and $y=y(u,v)$ be given implicitly as solutions of

$$\begin{cases} f_1 := x^2+x+y^2+u\cdot x - v = 0 \\ f_2 := y^2+y+x^2+v\,y - u = 0 \end{cases}$$

(actually, taken from the air!)

We need to check that x and y are well-defined and smooth near $(u,v)=(x,y)=0$ and compute the Jacobian at $(0,0)$

Solution:

1) Compute Jacobians

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y}\end{pmatrix} = \begin{pmatrix} 2x+1+u, & 2y \\ 2x, & 2y+1+v\end{pmatrix}$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)}\Big|_{(0,0)} = \begin{pmatrix} 1, & 0 \\ 0, & 1\end{pmatrix}$$ - invertable matrix

⇒ implicit function theorem works ⇒ $x(u,v)$ and $y(u,v)$ are smooth near $(0,0)$.

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial f_1}{\partial u}, & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u}, & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} x, & -1 \\ -1, & y \end{pmatrix}$$

$$\left.\frac{\partial(f_1,f_2)}{\partial(u,v)}\right|_{(0,0)} = \begin{pmatrix} 0, & -1 \\ -1, & 0 \end{pmatrix}$$

Thus,

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(f_1,f_2)}{\partial(x,y)}\right]^{-1} \cdot \frac{\partial(f_1,f_2)}{\partial(u,v)}$$ — chain Rule

And

$$\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(0,0)} = \begin{pmatrix} x'_u(0,0), & x'_v(0,0) \\ y'_u(0,0), & y'_v(0,0) \end{pmatrix} = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0, & -1 \\ -1, & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0, & -1 \\ -1, & 0 \end{pmatrix}$$

And, therefore,

$x'_u(0,0) = 0$, $x'_v(0,0) = -1$

$y'_u(0,0) = -1$, $y'_v(0,0) = 0$.

End of Part I (differential calculus of many variables).