Part II. Linear second order ODEs

Aim: to study the differential equation of the form:

(*) $$y'' + A(x)y' + B(x)y = Q(x)$$

x - scalar independent variable

$y = y(x)$ - unknown function (dependent variable)

$A(x), B(x), Q(x)$ - given functions.

Reminder: 1st order equations

(1) $$y' + A(x)y = B(x)$$ can be solved explicitly for all "reasonable" $A(x)$ and $B(x)$.

Indeed, multiplying the equation by $e^{\int A(x)dx}$,

$$\left(ye^{\int A(x)dx}\right)' = e^{\int A(x)dx} B(x)$$

which has a form

$$z' = Q_1(x) \qquad z := ye^{\int A(x)dx}$$

and can be easily integrated:

$$z(x) = z(0) + \int_0^x Q_1(y)dy$$

or

(**) $$y(x) = y(0)e^{-\int_0^x A(s)ds} + \int_0^x e^{-\int_s^x A(t)dt} \cdot B(s)ds$$

Formula for the general solution, depends on one arbitrary constant: $y(0) = y_0 \in \mathbb{R}$

If the initial condition $y(0) = y_0 \in \mathbb{R}$ is specified, then (**) gives a unique solution of (1) with that initial data.

$\Rightarrow$ The theory of the 1st order Linear ODEs is very simple and is completed by (**)!

Expect something similar for the 2nd order linear ODEs? Then, a surprise!

Nothing similar to (**) exists for 2nd order case!!!

For the "most" of equations (*), the solution cannot be written through integrals of A, B, and Q!!

$\Rightarrow$ Many special functions are introduced as solutions of the second order linear ODEs (Bessel, Airy, hypergeometric functions, etc.)

Example: The Airy function $Ai(x)$ solves

$$\boxed{y'' = xy}$$ - "simplest" 2nd order ODE with variable coefficients.

Can be written as an integral:

$$\boxed{Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt}$$

For more general equations (*) such formulas do not exist!

Natural question: What can we do with equation (*)? (if we cannot solve it explicitly)

Qualitative theory!
- existence of a solution
- uniqueness
- behaviour as $x \to \infty$: bounded/unbounded, oscillating/non-oscill.
- etc.

Q1: How to pose the initial data for equation (*)?

Only $\boxed{y(0) = y_0}$ (as for the 1st order equations is not enough $\Rightarrow$ one more condition is needed:

$$\boxed{y'(0) = y_1}$$!

Mechanical interpretation (see classical dynamics)

$\boxed{ma = F}$, $\boxed{y = y(t)}$ - position of an object at time moment t,

m - mass, $\boxed{F = F(t, y, y')}$ - force

$a = v'(t) = y''(t)$

which leads to

$$\boxed{m y'' = F(t, y, y')}$$ — 2nd order diff. equation:

We need to know the initial position $\boxed{y(0) = y_0}$ and the initial velocity $\boxed{y'(0) = y_1}$ in order to find the trajectory $y(t)$ = solve the second order equation.

Deep theorem from differential equations (will be studied later) guarantees that initial data $\boxed{y(0)=y_0,\ y'(0)=y_1}$ is enough for the existence and uniqueness of a solution.

Th. Let $A(x)$, $B(x)$ and $C(x)$ be continuous. Then, for every $\boxed{(y_0, y_1)\in\mathbb{R}^2}$ there is a unique solution $\boxed{y=y(x),\ x\in\mathbb{R}}$ of

$$\boxed{\begin{gathered} y''+A(x)y'+B(x)y=Q(x) \\ y(0)=y_0,\ y'(0)=y_1 \end{gathered}}$$

Some heuristics on how to find $y(x)$.
Assume that A, B, C are smooth. Try to find $y=y(x)$ using Taylor expansions:

$$\boxed{y(x)=y(0)+\frac{y'(0)}{1!}x+\frac{y''(0)}{2!}x^2+\cdots}$$

$\boxed{y(0)=y_0}$ and $\boxed{y'(0)=y_1}$ we know from the initial data
How to find $y''(0)$ – from the equation!

$$\boxed{y''(0)=Q(0)-A(0)y_1-B(0)y_0}$$

— " — $y'''(0)$? – Differentiating the equation!
By induction, we may find all $y^{(n)}(0)$ for all n (in a unique way!) $\Rightarrow$ all terms in

$$\boxed{y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(0)}{n!}x^n}$$

are determined in a unique way by y_0, y_1 and equation!
$\Rightarrow$ there exists a unique $y(x)$.
NOT a Rigorous proof! convergence?

Conclusion: (important) Every solution $y=y(x)$ of (*) is uniquelly determined by 2 numbers [(A, B, Q) are fixed!]
$y(0)$ and $y'(0)$
and, for every two numbers $(y_0, y_1)\in\mathbb{R}^2$ there is a solution
$\Rightarrow$ One-to-one correspondence between points of $\mathbb{R}^2$ and solutions.
$\Rightarrow$ $\boxed{\text{The space of all solutions of (*) is 2-dimensional} = \mathbb{R}^2}$

Important particular case homogeneous equation $C \equiv 0$:

(***) $\boxed{y'' + A(x)y' + B(x) = 0}$

Theorem (superposition principle) Let $y_1(x)$ and $y_2(x)$ be two solutions of (***). Then, for every $C_1, C_2 \in \mathbb{R}$

(S) $\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$

is again a solution of (***)

Proof: Just insert (S) to (***). □

⇒ The space of all solutions of (***) is a linear (vector) space.

What is the dimension of this solution space?

$\boxed{\dim V - ?}$

Answer: $\boxed{\dim V = 2}$! One-to-one linear correspondence between solutions and points of $\mathbb{R}^2$ is given by

$\boxed{y \mapsto (y(0), y'(0))}$

⇒ (important) $\boxed{\text{To solve (***)}} = \boxed{\text{to find 2 linear independent solutions } y_1(x) \text{ and } y_2(x)}$

Any other solution is a linear combination of y_1 and y_2

$$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)} \quad C_1, C_2 \in \mathbb{R}.$$

Remind:

$y_1(x)$ and $y_2(x)$ are linear independent iff

$C_1 y_1(x) + C_2 y_2(x) \equiv 0 \ \forall x$ for some C_1 and C_2 $\boxed{(C_1, C_2) \neq (0,0)}$ $= \boxed{\frac{y_1(x)}{y_2(x)} \equiv \text{const}}$

easy to check!

Assume now $y_1(x)$ and $y_2(x)$ be 2 linear independent solutions. Then, general solution is

(S) $\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$, $C_1, C_2 \in \mathbb{R}$

Need to satisfy the initial conditions:

$$y(0) = y_0, \quad y'(0) = y_1$$

$$\begin{cases} C_1 y_1(0) + C_2 y_2(0) = y_0 \\ C_1 y_1'(0) + C_2 y_2'(0) = y_1 \end{cases}$$

system on the unknowns C_1 and C_2

$$\begin{pmatrix} y_1(0), & y_2(0) \\ y_1'(0), & y_2'(0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

should have a unique solution for every $(y_0, y_1) \in \mathbb{R}^2$

Linear algebra $\Rightarrow$ $\det \begin{pmatrix} y_1(0), & y_2(0) \\ y_1'(0), & y_2'(0) \end{pmatrix} \neq 0$

Thus, we see that

$y_1(x), y_2(x)$ are linearly independent $\iff \det \begin{pmatrix} y_1(0), & y_2(0) \\ y_1'(0), & y_2'(0) \end{pmatrix} \neq 0$

The determiant has a special name Wronskian

$$W(x) = \det \begin{pmatrix} y_1(x), & y_2(x) \\ y_1'(x), & y_2'(x) \end{pmatrix} = y_1 y_2' - y_2 y_1'$$

and satisfies simple differential equation

(W) $\quad W'(x) = -A(x) \cdot W(x)$

Proof:

$$W'(x) = (y_1 y_2' - y_2 y_1')' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' =$$

(substitution $y_i'' = -A(x) y_i' - B(x) y_i$)

$$= y_1(-A y_2' - B y_2) - y_2(-A y_1' - B y_1) = -A(y_1 y_2' - y_2 y_1') = -A W \quad \square$$

(Looks as a magic, but there is a natural general theory behind - will be studied later)

$\Rightarrow$ Equation (W) can be easily solved (1st order linear ODE)

$$W(x) = e^{-\int_0^x A(s)ds} W(0)$$

⇒ We already know that there are not any algorithms to find the pair y_1 and y_2 of linear independent solutions of $\boxed{y'' + A(x)y' + B(x)y = 0}$

But imagine that one nontrivial solution $y_1(x)$ is somehow found (be known a-priori or guessed)

Then the second solution y_2 can be found algorithmically using the Wronskian. Indeed, we know

$$\boxed{y_1 \cdot y_2' - y_2 y_1' = W(x) = e^{-\int_0^x A(x)dx} \qquad W(0) = C \cdot e^{-\int_0^x A(t)dt}}$$

without loss of generality we may assume that $C = 1$ ($y_2 \to C y_2$ scaling)

That is the 1-st order ODE for $y_2(x)$ – can be explicitly solved! Namely, dividing it on y_1^2

$$\frac{y_1 \cdot y_2' - y_2 y_1'}{y_1^2} = \left(\frac{y_2}{y_1}\right)' = y_1^{-2}(x) \cdot e^{-\int_0^x A(t)dt}$$

OR

(2nd) $$\boxed{y_2(x) = y_1(x) \cdot \int_0^x [y_1(s)]^{-2} \cdot e^{-\int_0^s A(t)dt} ds}$$

– explicit formula for the second solution y_2.

Conclusions: for the homogeneous equation

$$\boxed{y'' + A(x)y' + B(x)y = 0}$$

1) The space of solutions in a 2-dim vector space.

2) If two partial solutions $y_1(x)$ and $y_2(x)$ are known, the general solution is a linear combination

(*) $$\boxed{y(x) = C_1 y_1(x) + C_2 y_2(x)}$$

3) The solution satisfying the initial conditions $\boxed{\begin{matrix} y(0) = y_0 \\ y'(0) = y_1 \end{matrix}}$ can be found from (*) solving the system

$$\boxed{\begin{pmatrix} y_1(0), & y_2(0) \\ y_1'(0), & y_2'(0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}$$

4) There is no general way to find y_1 and y_2, but if y_1 is known the second solution y_2 can be found using the Wronskian, see formula (2nd).

Big Example: constant coefficients

$$y'' + Ay' + By = 0, \quad A, B \in \mathbb{R}$$ - independent of x

Need to find 2 linear independent solutions y_1 and y_2. Try the simplest - exponents

$$y(x) = e^{\lambda x}$$ λ is unknown number to be detected

$$y'(x) = \lambda y(x)$$
$$y''(x) = \lambda^2 y(x) \Rightarrow (\lambda^2 + A\lambda + B)e^{\lambda x} = 0$$

Non-trivial solution $e^{\lambda x}$ $\Leftrightarrow$ $\lambda^2 + A\lambda + B = 0$ (Q)

Characteristic-equation

Equation (Q) is a quadratic equation $\Rightarrow$ has two roots

$$\lambda_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$ may be complex

If $\lambda_1 \neq \lambda_2$ $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ - two linearly independent solutions and

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$ is a general solution

$$D = A^2 - 4B \neq 0$$

In particular, if $D = A^2 - 4B > 0$ - two Real exponents (non-oscillatory solution)

if $D = A^2 - 4B < 0$ - two complex conjugate roots

$$\lambda_1 = \alpha + i\omega$$
$$\lambda_2 = \alpha - i\omega$$

$\alpha = -\frac{A}{2}$, $\omega = \sqrt{4B - A^2}$

$$y_1(x) = e^{(\alpha + i\omega)x}, \quad y_2(x) = e^{(\alpha - i\omega)x}$$

Using Moivre formula $e^{i\omega x} = \cos \omega x + i \sin \omega x$

We may construct two real-valued solutions

$$y_1(x) = e^{\alpha x} \cos \omega x, \quad y_2(x) = e^{\alpha x} \sin \omega x$$

(oscillatory case)

$$y(x) = e^{\alpha x}(C_1 \cos \omega x + C_2 \sin \omega x)$$ general solution.

Need to understand what to do if

Resonant case $\lambda_1 = \lambda_2 = -A/2$ $\quad D = A^2 - 4B = 0$

Then, we have only one solution $y_1(x) = e^{-\frac{A}{2}x}$

How to find the second one? For instance, using the Wronskian formula (2nd)

$$y_2(x) = e^{-\frac{A}{2}x} \cdot \int_0^x \left[e^{-\frac{A}{2}s}\right]^{-2} \cdot e^{-As}\,ds = x\,e^{-\frac{A}{2}x}$$

$$y(x) = e^{-\frac{A}{2}x}(C_1 + C_2 x)$$ - general solution for that case.

Examples:

1) $y'' - y = 0$, $\Rightarrow$ $\lambda^2 - 1 = 0$ $\quad \lambda = \pm 1$

$y(x) = C_1 e^x + C_2 e^{-x}$ - general solution

2) $y'' + y = 0$ $\quad \lambda^2 + 1 = 0$, $\quad \lambda = \pm i$

$y(x) = C_1 e^{ix} + C_2 e^{-ix} = \tilde{C}_1 \cos x + \tilde{C}_2 \sin x$ - general solution

3) $y'' - 2y' + y = 0$ $\quad \lambda^2 - 2\lambda + 1 = 0$ $\quad (\lambda - 1)^2 = 0$

$\lambda_1 = \lambda_2 = 1$

$y(x) = (C_1 + C_2 x)e^x$

Non-homogeneous equations.

(*) $\quad y'' + A(x)y' + B(x)y = Q(x)$

Key idea:

general solution of non-homogeneous equation = general solution of homogeneous equation + partial solution of non-homogeneous equation

Proof: Let $y_*(x)$ be a fixed (partial) solution of (*) and let $y(x)$ be any other solution $\Rightarrow$

$Z(x) := y(x) - y_*(x)$ - solves homogeneous equation.

$\Rightarrow$ if $y_1(x)$ and $y_2(x)$ - two linear independent solutions of the homogeneous equation and $y_*(x)$ - some (partial) solution of the non-homogeneous equation. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_*(x)$$

general solution of the non-homogeneous equation

Assume that we know $y_1(x)$ and $y_2(x)$ (already discussed how to find them)

Problem: How to find a single solution of the non-homogeneous equation?

Solution (variation of constants method)

Try to find $\boxed{y_*(x) = C_1(x)\,y_1(x) + C_2(x)\,y_2(x)}$

where C_1 and C_2 no more integration constants but unknown functions to be found.

Two unknown functions and only one equation (*)

$\Rightarrow$ We may add one more equation (in a most convenient way)

$$y_*'(x) = C_1(x)\,y_1'(x) + C_2(x)\,y_2'(x) + \boxed{\left[C_1'(x)\,y_1(x) + C_2'(x)\,y_2(x)\right]}$$

"Bad" term which will produce the second derivatives of C_i

$\Rightarrow$ Let us "kill" it adding the second equation

$$\boxed{C_1'(x)\,y_1(x) + C_2'\,y_2(x) \equiv 0}$$

Then

$$\boxed{\begin{aligned} y_*' &= C_1\,y_1' + C_2\,y_2' \\ y_*'' &= C_1\,y_1'' + C_2\,y_2'' + C_1'\,y_1' + C_2'\,y_2' \end{aligned}}$$

Inserting that to (*), only $C_1'\,y_1' + C_2'\,y_2'$ will survive in the left-hand side (y_1 and y_2 solves the homogeneous problem) and we have

$$\boxed{\begin{aligned} C_1'(x)\,y_1(x) + C_2'(x)\,y_2(x) &= 0 \\ C_1'(x)\,y_1'(x) + C_2'(x)\,y_2'(x) &= Q(x) \end{aligned}}$$

$$\boxed{\begin{pmatrix} y_1(x), & y_2(x) \\ y_1'(x), & y_2'(x) \end{pmatrix}\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ Q \end{pmatrix}}$$

The determinant

$$\boxed{W(x) = \det\begin{pmatrix} y_1(x), & y_2(x) \\ y_1'(x), & y_2'(x) \end{pmatrix} \neq 0}$$ (Wronskian)

(We assume y_1 and y_2 be linear independent)

$\Rightarrow$ We can find

$$\boxed{\begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} y_1(x), & y_2(x) \\ y_1'(x), & y_2'(x) \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ Q(x) \end{pmatrix}}$$

and then find $C_1(x)$ and $C_2(x)$ by integration

Big example: constant coefficients and

$$Q(x) = \mathrm{Re}\left(\sum a_k \cdot e^{\mu_k x} \cdot x^{n_k} \right)$$

$\Rightarrow$ Right-hand side contains only terms

$$x^n e^{\alpha x},\ x^m,\ x^n e^{\alpha x}\cos\beta x,\ x^n e^{\alpha x}\sin\beta x$$

and their sums.

Problem: to find a partial solution (can be done by variation of constants), but it is often simpler to find it using undetermined coefficients method and the following **Rules:**

(1) If $Q(x) = e^{\mu x} P_n(x)$ and $\mu^2 + A\mu + B \neq 0$ (non-resonant case)

$\mu \in \mathbb{C}$, P_n - polynomial of order n with complex coefficients

$\Rightarrow$ Partial solution can be found in the form

$$e^{\mu x}\tilde{P}_n(x), \quad \tilde{P}_n(x) = a_n x^n + \dots + a_0$$

$\tilde{P}_n(x)$ - is a polynomial of the same order with unknown coefficients to be determined.

(2) If $\mu^2 + A\mu + B = 0$ - single root (single resonance)

Partial solution: $e^{\mu x}\tilde{P}_{n+1}(x)$, $\tilde{P}_{n+1}(x)$ - polynomial of order $(n+1)$ with unknown coefficients to be determined

(3) If $\mu^2 + A\mu + B = 0$ - double root: $\lambda_1 = \lambda_2 = \mu$ (double resonance)

Partial solution: $e^{\mu x}\tilde{P}_{n+2}(x)$, $\tilde{P}_{n+2}(x)$ - polynomial of order $(n+2)$ with unknown coefficients.

Can be checked using the variation of constants.

NB: Do not forget that $\mu = \alpha + i\omega$ can be a complex number. The real-valued form of that term is

$$e^{\alpha x}\left(P_n(x)\cos\omega x + Q_n(x)\sin\omega x \right)$$

P_n, Q_n - real-valued polynomials of order n.

Partial solution: $e^{\alpha x}\left(\tilde{P}_n(x)\cos\omega x + \tilde{Q}_n(x)\sin\omega x \right)$

OR $(\tilde{P}_{n+1}, \tilde{Q}_{n+1})$ OR $(\tilde{P}_{n+2}, \tilde{Q}_{n+2})$ in resonant cases.

Important: Even if $Q_n \equiv 0$ you need $\tilde{Q}_n(x)$ of order n in the formula for the partial solution!

(often mistake is to forget this!)

Examples:

① $y'' + y = e^x(x+1)$ $\quad m = 1$, $m^2+1 \neq 0$

$\Rightarrow$ non-resonant case

$y_*(x) = e^x(ax+b)$, $y_*' = e^x(ax+b) + e^x \cdot a = e^x(ax+(a+b))$

$y_*'' = e^x(ax+(2a+b))$

$y_*'' + y_* = e^x(2ax+2a+2b) = e^x(x+1)$

$\Rightarrow$ $2a = 1$, $2a+2b = 1$ $\Rightarrow$ $a = 1/2$, $b = 0$

$y_*(x) = \frac{1}{2} x e^x$

General solution:

$y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2} x e^x$

② $y'' + w^2 y = \sin wx$ $\quad m = iw$ $\quad m^2 + w^2 = 0$

$\alpha_1 \neq \alpha_2$ – single resonance

$y_*(x) = (ax+b)\sin wx + (cx+d)\cos wx$

these terms can be omitted since give the solution of a homogeneous equation

Thus,

$y_*(x) = x(a \sin wx + b \cos wx)$

$y_*'(x) = a \sin wx + b \cos wx + x(aw \cos wx - bw \sin wx)$

$y_*''(x) = 2aw \cos wx - 2bw \sin wx + x(-aw^2 \sin wx - bw^2 \cos wx)$

$y_*'' + w^2 y_* = 2aw \cos wx - 2bw \sin wx = \sin wx$

$a = 0$, $b = -1/2w$

$y_*(x) = -\frac{1}{2w} x \cos wx$

General solution

$y(x) = C_1 \cos wx + C_2 \sin wx - \frac{1}{2w} \cdot x \cdot \cos wx$