Part III Integration.

Lecture 16. Riemann integration: an extended reminder.

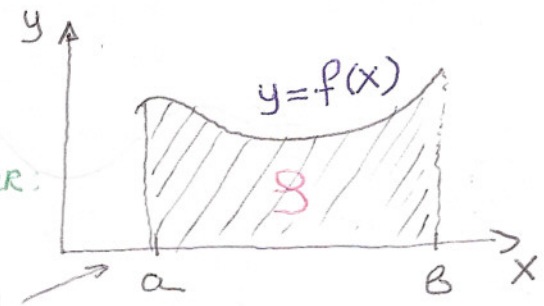
Aims

- to clarify the relations between integrals, integral sums and areas.
- to understand how to deduce the main properties of integrals using integral sums.

Global aim: Extend that to multi-dimensional case.

Riemann integral: $f: [a, B] \to \mathbb{R}$ - a scalar function

$$\int_a^B f(x)\,dx - ?$$



What is that? Standard answer:

it is the area of the region bounded by the graph $y = y(x)$

What is "bad" with this answer?

Why is it incomplete?

Problem: What is the area S???

Indeed, we know how to compute the areas of simple figures (rectangles, triangles, etc.) or most general figures which can be built by the finite numer of triangles (and disks).

But what is the area of a region restricted by the graph of an arbitrary (smooth, but not linear) function?

Most natural answer: it is an integral!

Tautology! Something else is needed!

Nothing is actually defined!!!

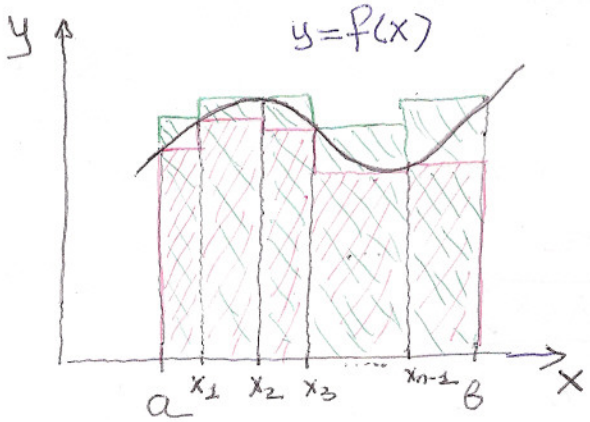
(although very transparent tautology!)

Main question: How to define (compute) the area of a non-linear figure?
= How to define the integral?

Key idea: To approximate the nonlinear figure by simple ones (constructed by finitely many rectangles or/and triangles) and pass to the limit.

Example: appxoximation of a circle by n-sided right polygones and finding the area of a disk by passing to the limit $\boxed{n \to \infty}$.

Realization:



Consider a partition $[a,b]$

$$\boxed{P := a = x_0 < x_1 < \cdots < x_n = b}$$

or $\boxed{n \gg 1}$ subsegments of length $\boxed{\delta x_i = x_i - x_{i-1}}$

And let

$$\boxed{\max_i f = \max_{x \in [x_{i-1}, x_i]} f}$$

$$\boxed{\min_i f = \min_{x \in [x_{i-1}, x_i]} f}$$

Then, "obviously"

$$\boxed{S_{red} \leq S \leq S_{green}}$$

and we can compute S_{red}, S_{green} (simple figures!)

$$\boxed{\sum_{i=1}^{n} \min_i f \cdot \delta x_i \leq S \leq \sum_{i=1}^{n} \max_i f \cdot \delta x_i}$$

$$\sum_{i=1}^{n} \min_i f \cdot \delta x_i = S_{red} \qquad \sum_{i=1}^{n} \max_i f \cdot \delta x_i = S_{green}$$

S_{red} and S_{green} are called lower and upper Daurboux sums (will be studied in details on Real Analysis II)

$\Rightarrow$ approximation

$$\boxed{\sum_{i=1}^{n} \min_i f_i \cdot \delta x_i \leq \int_a^b f(x)\,dx \leq \sum_{i=1}^{n} \max_i f \cdot \delta x_i}$$

(often used in numerics to compute the integral)

Approximation becomes better if we refine the partition = make smaller the mesh $\delta P = \max_i \delta x_i$ of the partition P.

Why? Consider an equidistant partition P on n-equal segments $\delta P = \delta x_i = \frac{b-a}{n}$ and let function f be smooth (at least $f'(x)$ be bounded). Then, by the mean value theorem

$$\max_i f - \min_i f = f(x_+) - f(x_-) = f'(\tilde{t})(x_+ - x_-) \le M \cdot \frac{b-a}{n}$$

since $x_+, x_- \in [x_{i-1}, x_i]$ and $M = \max f'$

Thus

$$S_{\text{green}} - S_{\text{red}} = \sum_{i=1}^{n} (\max_i f - \min_i f) \cdot \frac{b-a}{n} \le$$

$$\le \sum_{i=1}^{n} M \cdot \frac{b-a}{n} \cdot \frac{b-a}{n} = M \frac{(b-a)^2}{n} \xrightarrow{n\to\infty} 0$$

Thus, in the limit $\delta P \to 0$ both S_{green} and S_{red} tend to the same number which is by def an integral:

$$\int_a^b f(x)\,dx = \lim_{\max \delta x_i \to 0} \sum_{i=1}^{n} \max_i f \, \delta x_i = \lim_{\max \delta x_i \to 0} \sum \min_i f \, \delta x_i$$

The desired definition of the integral!

One more observation: if the integral sums with $\max_i f$ and with $\min_i f$ converge to the same limit, we may replace them by the value $f(t_i)$ at any point $t_i \in [x_{i-1}, x_i]$

$$\int_a^b f(x)\,dx := \lim_{\max \delta x_i \to 0} \sum_{i=1}^{n} f(t_i) \delta x_i$$

= Standard def, roughly

Integral = limit of integral sums

Remark: Integral is well-defined NOT only for smooth, but also for all continuous functions (the proof is harder) and for many discontinuous ones. See Real Analysis II (in future!)

Conclusion: Integral is defined as a limit of integral sums. The formula

$$\boxed{\text{Integral}} = \boxed{\text{area restricted by the graph}}$$

defines the area in a rigorous way, NOT the integral !!!

(and, of course, is a good visalization of the concept)

How to work with this def? will be used for multiple integrals as well!

Example I: Newton-Leibniz formula (fundamental theorem of calculus)

If $f \in [a,b]$ is continuous and $F(x)$ is anti-derivative $\boxed{F'(x) = f(x)}$

Then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Idea of the proof: Need to estimate/compute the integral sum

$$\boxed{I_n = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})}$$

say, for the equidistant partition with $\boxed{t_j = x_{j-1}}$ - left point of the interval.

Where we can see terms like

$$f(x_{i-1})\,\delta x_i \; - ?$$

= linear part of Taylor expansions

$$\boxed{F(x_i) - F(x_{i-1}) = f(x_{i-1})\delta x_i + O(\delta x_i^2)}$$

if we neglect the H.O.T., we obtain

$$\boxed{I_n \cong \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(x_n) - F(x_0) = F(b) - F(a)}$$

telescopic series!

The error $O(\delta x_i^2)$ tends to zero as $\delta x_i \to 0$ and in the limit we have exact f-la.

Even simpler: By mean value theorem

$$\boxed{F(x_i) - F(x_{i-1}) = f(t_i)\delta x_i}, \quad t_i \in [x_{i-1}, x_i]$$

$$\Rightarrow \quad \boxed{\sum_{i=1}^{n} f(t_i)\delta x_i = F(b) - F(a)}$$

an exact f-la for that choice of t_i

Passing to the limit $\boxed{\max \delta x_i \to 0} \Rightarrow$ Newton-Leibniz f-la.

□

Example II: Change of variables

Can be obtained from Newton-Leibniz and the chain rule, but we want to do that directly. (will be used for the multiple integrals where we do not have Newton-Leibniz f-la)

Let $f: [a,b] \to \mathbb{R}$ be continuous and let

$\varphi: [c,d] \leftrightarrow [a,b]$ be a smooth one-to-one map

$\varphi(c)=a$, $\varphi(d)=b$

$\varphi'(t) \geq 0$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_c^d f(\varphi(t))\,\varphi'(t)\,dt$$

Idea of the proof:

Consider a partition $c=t_0<t_1 \cdots < t_n=d$ for the right integral with the corresponding integral sum

$$I_R = \sum_{i=1}^{n-1} f(\varphi(t_{i-1}))\,\varphi'(t_{i-1})\,\delta t_i$$

This partition generates a partition for the left integral $a=x_0<x_1 \cdots < x_n=b$ by $x_i=\varphi(t_i)$

with the integral sum for the left integral

$$I_L = \sum_{i=1}^{n-1} f(x_{i-1})\,\delta x_i = \sum_{i=1}^{n-1} f(\varphi(t_{i-1}))\,(\varphi(t_i)-\varphi(t_{i-1}))$$

Need to compare I_L and I_R and to show that the difference is neglectable if the mesh of the partition is small. To do that, we write Taylor expansions:

$$\delta x_i = x_i - x_{i-1} = \varphi(t_i)-\varphi(t_{i-1}) = \varphi'(t_{i-1})\,\delta t_i + O(\delta t_i^2)$$

$\Rightarrow$ At the leading term $\delta x_i \approx \varphi'(t_{i-1})\,\delta t_i$ + the higher order corrector which will disappear in the limit. $\square$.

We will use this example in order to find the formulas for double and triple integrals

ALL reasons will be very similar!