Lecture 17-19. Multiple integrals.

Aims:

- learn the defs for double and triple integrals.
- understand how to reduce the multiple integrals to the iterated ones (Fubini theorem)
- learn how to change variables in multiple integrals
- consider the examples of polar, cylindrical and spherical coordinates.

1D case

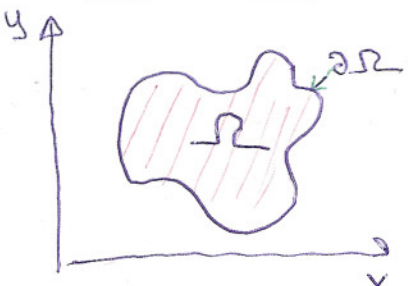
$$\int_a^b f(x)\,dx = \lim_{\delta x_i \to 0} \sum_{i=1}^{n} f(t_i)\,\delta x_i$$

$a = x_0 < x_1 < \dots < x_n = b$

$\delta x_i = x_i - x_{i-1}$

$t_i \in [x_{i-1}, x_i]$

2D case: Let $\Omega \subset \mathbb{R}^2$ be a region of the plane restricted by a "good" curve $\partial\Omega$ - boundary of Ω (smooth or piece-wise smooth)



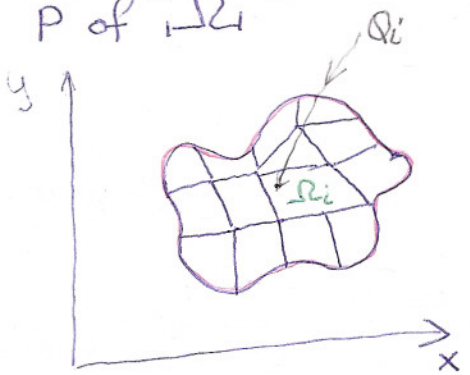
(analogue of a segment $[a,b]$ for 2D case)

Let also $f: \mathbb{R}^2 \to \mathbb{R}$ or $f: \Omega \to \mathbb{R}$ be a "reasonably smooth" function

Want to define and compute: the DOUBLE integral

$$\iint_\Omega f(x,y)\,dx\,dy - ?$$

Analogously to the 1D case, consider a partition P of Ω



$$\Omega = \bigcup_{i=1}^{n} \Omega_i$$ disjoint union

Ω_i - reasonably smooth small subregions

$\max \operatorname{diam} \Omega_i \ll 1$

For every Ω_i fix a point $(x_i, y_i) = Q_i$ in Ω_i and consider the associated

$\delta S_i := \text{AREA}(\Omega_i)$

Riemann sum $\sum_{i=1}^{n} f(Q_i)\,\delta S_i$

The analogue of length in 2D is an area!

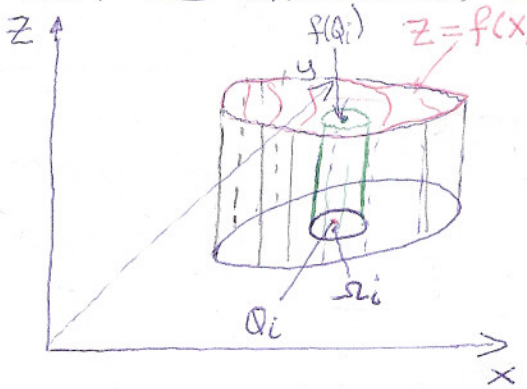
Analogously to 1D, we may define

$$\iint_{\Omega} f(x,y)\,dx\,dy := \lim_{\max \operatorname{diam}(\Omega_i)\to 0} \sum_{i=1}^{n} f(Q_i)\delta S_i$$

In a complete analogy with 1D case, we can introduce upper and lower Darboux sums (with $\max\limits_{\Omega_i} f$ and $\min\limits_{\Omega_i} f$) and show that, for reasonably smooth f and $\partial\Omega$ both of them tend to the same number (integral) when $\operatorname{diam}(\Omega_i)\to 0$ and the limit exists

(The rigorous theory exists, but unlikely be studied at the undegraduate level!)

Moreover, the 1D analogy with an area under the graph can be extended to the 2D case



$z=f(x,y)$ - is now a surface

$f(Q_i)\delta S_i$ is the volume of a cylinder generated by Ω_i (of length $f(Q_i)$) and the Riemann sum $\sum f(Q_i)\delta S_i$ approximates the volume restricted by Ω and the surface $z=f(x,y)$

Thus

$$\text{Double integral} = \text{Volume under the surface } z=f(x,y)$$

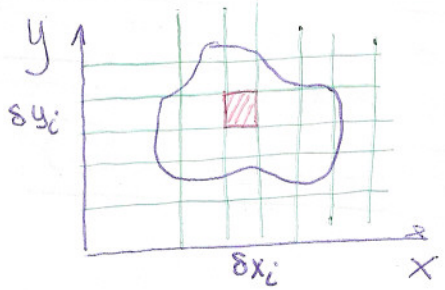
And, again, this defines the volume, not the integral! (the only difference is that it's much more difficult to plot a good picture!!!)

Different notations:

$$\iint_{\Omega} f\,dx\,dy = \int_{\Omega} f\,dx\,dy$$

$dx\,dy$ is used since for the most natural Rectangular partitions

$$\delta S_i = \delta x_i \cdot \delta y_i$$



(a single sign of the integral is often used instead of the doubled one)

Do not be confused!!

Another equivalent notation:

$$\iint_\Omega f\,dS \quad \text{or} \quad \int_\Omega f(x,y)\,dS$$

dS = "infinitesimal area" (area differential)

More advanced notation:

$$\int_\Omega f(x,y)\,dx\wedge dy$$

($\wedge$) is a wedge product

(Roughly speaking, means the area again but in the language of differential geometry)

Do not be confused! All notations mean the same: Integral is ALWAYS a limit of the proper integral sums!!

The 3D case: completely analogous, but now Ω is a region of a 3D space

$$f = f(x,y,z) : \Omega \to \mathbb{R}$$

$\Omega = \bigcup \Omega_i$ disjoint partition

The only difference is that now

$$\delta V_i = \text{Volume}(\Omega_i)$$

(The 3D analogue of an area is a volume!)

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz := \lim_{\text{diam}(\Omega_i)\to 0} \sum f(Q_i)\,\delta V_i$$

(will be briefly considered below)

Again, equivalent notations

$$\iiint_\Omega f\,dx\,dy\,dz = \int_\Omega f\,dx\,dy\,dz = \int_\Omega f\,dV = \int_\Omega f\,dx\wedge dy\wedge dz$$

(volume)

The nD case ($n>3$): completely analogous will be NOT considered here!

Thus, the multiple integrals are now defined = we understood what is that.

Next natural question:

How to work with them / how to compute them?

(Extremely difficult to compute them via partitions and integral sums (if you are not a computer, of course!), even in the 1D case).

Want to reduce multiple integrals to the single one (which we already know how to compute!)

Answer is given by the Fubini theorem.

Start with 2D case + rectangular domain

Theorem Let $\Omega = \underset{x}{[a,b]} \times \underset{y}{[c,d]}$ and f is "reasonable" (continuous in Ω is enough, but works for many discontinuous f as well)

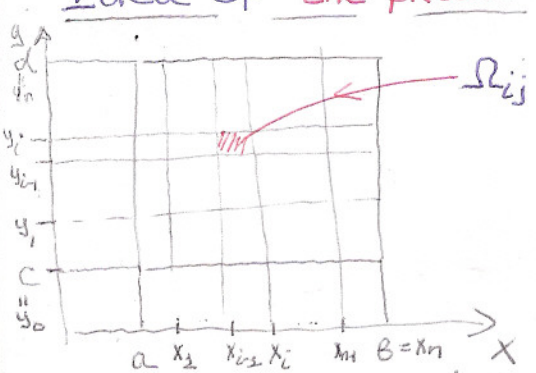
Then

$$\iint_\Omega f(x,y)\,d\Omega = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$

first integrated by y then by x — iterated integrals — inversed order ist by x then by y

Reduces the problem to calculating only the single integrals (Calculus I)

Idea of the proof:



consider the rectangular partition P_n of Ω generated by the partitions

$$a = x_0 < x_1 < \dots < x_n = b$$

and

$$c = y_0 < y_1 < \dots < y_n = d$$

of segments $[a,b]$ and $[c,d]$.

Naturally parametrized by 2 numbers i and j

$$\Omega_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

$$\delta\Omega_{ij} = \delta x_i \cdot \delta y_j$$

We just take $Q_{ij} = (x_i, y_j) \in \Omega_{ij}$

And the integal sum reads

$$I(P_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\,\delta x_i\,\delta y_j$$

This sum can be written as

$$I(P_n) = \sum_{i=1}^{n_1} \delta x_i \left(\sum_{j=1}^{n_2} f(x_i, y_j)\, \delta y_j \right)$$

← integral sum for $\int_c^d f(x_i, y)\,dy$

$\Rightarrow$ We expect

$$I(P_n) = \sum_{i=1}^{n_1} \delta x_i \cdot \int_c^d f(x_i, y)\,dy + \text{neglectable terms}$$

H.O.T.

But that is an integral sum for $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$

$\Rightarrow$ Passing to the limit $\boxed{\operatorname{diam}(\Omega_{ij}) \to 0}$ gives the desired formula $\square$.

Examples:

a) $$\iint_{[0,1]\times[1,2]} xy\,dx\,dy = \int_0^1 \left(\int_1^2 xy\,dy \right) dx =$$

$$= \int_0^1 x \frac{y^2}{2}\Big|_{y=1}^{y=2} dx = \int_0^1 \frac{3}{2} x\,dx = 3/4$$

b) $$\iint_{[a,b]\times[c,d]} f(x)\cdot g(y)\,dx\,dy = \int_a^b \left(\int_c^d f(x)g(y)\,dy \right) dx =$$

$$= \int_a^b f(x) \left(\int_c^d g(y)\,dy \right) dx = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

(Useful formula, works for rectangular domains only!)

The 3D case: Briefly $\Omega = [a,b]\times[c,d]\times[e,f]$ (x) (y) (z)

$$\iiint_\Omega f(x,y,z)\,dz = \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz \right) dy \right) dx$$

+ ALL 6 different orders of integration give the same result!

The nD case is analogous.

Thus, for the rectangular domains the problem is solved.

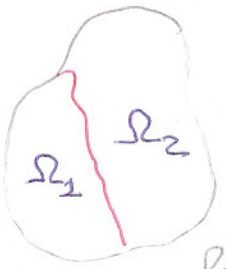
NB (very important): Unlike the 1D case, NO "ANTIDERIVATIVES" for multiple integrals! $\Rightarrow$ Should be computed for every Ω separately.

computing multiple integrals without taking care about the domain ("indefinite integration") is a NONSENSE !!!

General non-rectangular regions of integration:

Before, some elementary, but useful properties:

1) **Additivity**: if $\Omega = \Omega_1 \cup \Omega_2$ disjoint union



$$\iint_{\Omega_1 \cup \Omega_2} f\,d\sigma = \iint_{\Omega_1} f\,d\sigma + \iint_{\Omega_2} f\,d\sigma$$

Analogue of $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$ for the multi-dimensional case.

2) **Linearity**:

$$\iint_{\Omega} (\alpha f_1 + \beta f_2)\,d\sigma = \alpha \iint_{\Omega} f_1\,d\sigma + \beta \iint_{\Omega} f_2\,d\sigma$$

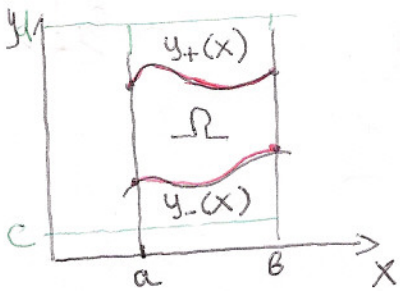
f_1, f_2 - functions, α, β - constants

3) **Positivity**: if $f \geq 0$ everywhere, then

$$\iint_{\Omega} f\,d\sigma \geq 0$$

(ALL follow from the integral sums).

Consider now the region Ω restricted by 2 curves $y_+(x)$ and $y_-(x)$



Fubini theorem for that case:

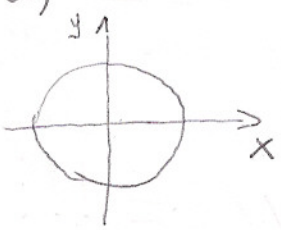
$$\iint_{\Omega} f(x,y)\,d\sigma = \int_a^b \left(\int_{y_-(x)}^{y_+(x)} f(x,y)\,dy \right) dx$$

can be reduced to the rectangular case just extending f by ZERO outside of Ω.

Together with additivity allows to reduce double integrals to iterated ones in a general case.

Examples:

a) Ω = disk $x^2 + y^2 = 1$



$y_+ = +\sqrt{1-x^2}$

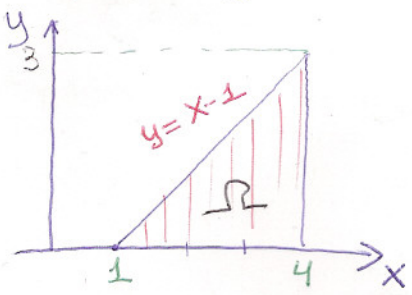
$y_- = -\sqrt{1-x^2}$

$$\iint_{\Omega} f\,d\sigma = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} f(x,y)\,dy \right) dx$$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f(x,y)\,dx \right) dy$$

in the reversed order

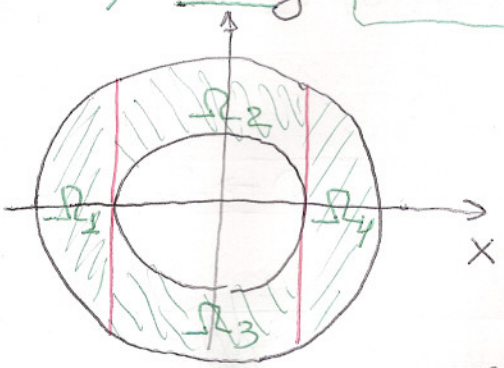
b) Triangle:



$$\iint_\Omega f(x,y)\,d\Omega = \int_1^4 \left(\int_0^{x-1} f(x,y)\,dy \right) dx$$

$$= \int_0^3 \left(\int_{y+1}^4 f(x,y)\,dy \right) dx$$

c) Ring $\Omega = \{ 1 \le x^2+y^2 \le 4 \}$. Should be splitted on 4 sub-regions



$$\iint_\Omega f\,d\Omega = \iint_{\Omega_1} f\,d\Omega + \iint_{\Omega_2} f\,d\Omega + \iint_{\Omega_3} f\,d\Omega + \iint_{\Omega_4} f\,d\Omega$$

And in every of that 4 regions we can present the double integral as the iterated one, e.g.,

$$\iint_{\Omega_2} f\,d\Omega = \int_{-1}^{1} \left(\int_{+\sqrt{1-x^2}}^{+\sqrt{4-x^2}} f(x,y)\,dy \right) dx$$

d) 3D Ball $\Omega = \{ x^2+y^2+z^2 \le 1 \}$

$$\iiint_\Omega f\,dV = \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \left(\int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz \right) dy \right) dx$$

Want the order
1st (z)
2nd (y)
3rd (x)

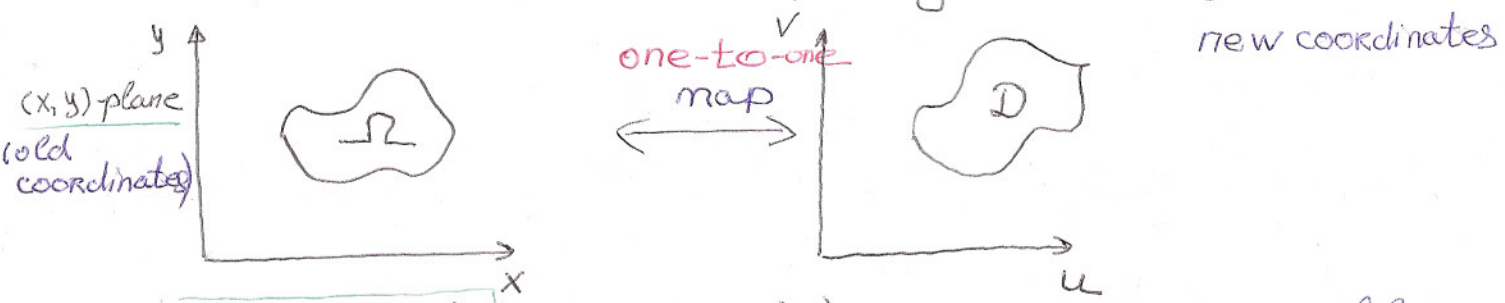
1) (x) - varies from (-1) to (1)
2) For x-fixed (y) varies from $-\sqrt{1-x^2}$ to $+\sqrt{1-x^2}$
3) For (x,y)-fixed (z) varies from $-\sqrt{1-x^2-y^2}$ to $+\sqrt{1-x^2-y^2}$

Practical advises:

1. Always plot a correct picture of the region Ω. (at least in 2D case)
2. 1st determine the limits of integration and only after that start to compute integrals (!)
3. Split the initial region into parts where the f-theorem works (if necessary), see example c).
4. If the order 1st (y) 2nd (x) is required, at first find where (x) varies and then, for fixed (x) find the limits for (y)

Change of variables in multiple integrals:

(2nd important tool for computing the integrals)



$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$$ - change of variables

$$\Omega = T(D), \quad D = T^{-1}(\Omega)$$

Theorem: Let $f: \Omega \to \mathbb{R}$ and let $T: D \to \Omega$ be smooth and one-to-one. Then

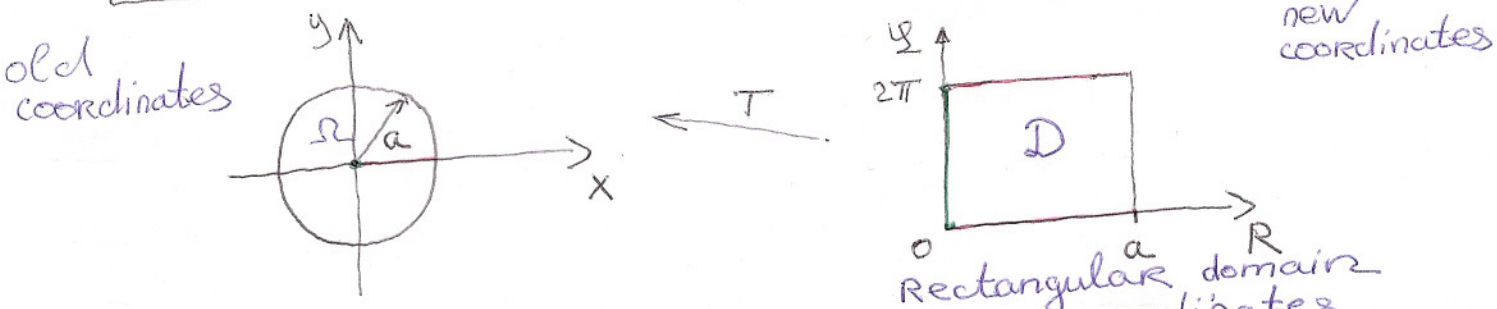
$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_{D} f(x(u,v), y(u,v)) \, |\det J(T)| \, du\,dv$$

$|\det J(T)|$ modulus of the det of Jacobi matrix = Jacobian.

Example: Circular domains and polar coordinates.

$$\Omega = \{ x^2 + y^2 \le a^2 \}$$

$$x = x(R, \varphi) = R\cos\varphi$$
$$y = y(R, \varphi) = R\sin\varphi$$



Not entirely one-to-one
green line is mapped to the origin (single point) and two red lines is mapped to a single line.
Theory still work
(the non-one-to-one set is "neglectable")

$$J = \begin{pmatrix} \cos\varphi & -R\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\det J = R$$

Final answer:

$$\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{D} f(R\cos\varphi, R\sin\varphi)\,R\,dR\,d\varphi =$$

$$= \int_0^a R\left(\int_0^{2\pi} f(R\cos\varphi, R\sin\varphi)\,d\varphi\right)dR$$

In particular, the area of Ω:

$$\mathcal{S} = \iint_{\Omega} 1\cdot dx\,dy = \int_0^a R\int_0^{2\pi} 1\,d\varphi\,dR = 2\pi\int_0^a R\,dR = \pi a^2$$

(area of a disk!!!)

Symmetric functions $f(x,y) = f(\sqrt{x^2+y^2})$

$$\iint_{\Omega} f(x,y)\,dx\,dy = 2\pi\int_0^a R\,f(R)\,dR$$

Idea of the proof (BRIEFLY): How do the Jacobian come into play?

Consider the rectangular partition P of D

$D = \bigcup_{i,j} D_{ij}$, $D_{i,j} = [u_{i-1}, u_i]\times[v_{i-1}, v_i]$ $Q_{ij} = (x_i, y_j)$

$T: D \to \Omega$ is one-to-one P generates a partition of Ω: $T(P)$ $\Omega = \bigcup_{i,j}\Omega_{ij}$, $\Omega_{ij} = T(D_{ij})$

$$\iint_{\Omega} f\,dx\,dy \sim \sum_{i,j}{}' f(T(Q_{ij}))\,\mathrm{AREA}(\Omega_{ij})$$

We know that $\mathrm{AREA}(D_{ij}) = \delta u_i\,\delta v_j$

Need to find $\mathrm{AREA}(\Omega_{ij})$?, $\Omega_{ij} = T(D_{ij})$ non-linear figure (non-linear image of a rectangle D_{ij})

But D_{ij} is very small and T is smooth $\Rightarrow$ T is almost linear on D_{ij} (up to the neglectable H.O.T.)

$\Rightarrow$ can be replaced by Linear map

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = J(T)\big|_{Q_{ij}} \begin{pmatrix} du \\ dv \end{pmatrix}$$

The image of a rectange D_{ij} under the LINEAR map with matrix J is a parallellogram generated by vectors $(x'_u, y'_u)\cdot\delta u_i$ and $(x'_v, y'_v)\cdot\delta v_j$

$$\mathrm{AREA} = \det\begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}\delta u_i\,\delta v_j = \det J\cdot\delta u_i\,\delta v_j$$

Linear algebra!

□

3D case: (BRIEFLY)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix}$$

old variables; new variables

one-to-one change of variables

$$\Omega = T(D), \quad D = T^{-1}(\Omega)$$

Theorem (analogously to 2D case) $f: \Omega \to \mathbb{R}$ "good" function

T is smooth and one-to-one

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{D} f(x(\cdot), y(\cdot), z(\cdot)) \det J \cdot du\,dv\,dw$$

Again the Jacobian appears as a factor (by very similar reasons)

Examples: 2 standard non-Euclidean coordinates in 3D

1) Cylindrical coordinates (useful for the cylinder-like domains

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\begin{pmatrix} R \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

$$x(R,\varphi,z) = R\cos\varphi$$
$$y(R,\varphi,z) = R\sin\varphi$$
$$z(R,\varphi,z) = z$$

= polar coordinates in the (x,y)-plane and the z-coordinate remains unchanged.

TRANSFORMS: a circular cylinder $\Omega = \{z \in [0,a],\ x^2+y^2 \le b^2\}$ to the rectangular parallelepiped

$$D = [0,b] \times [0,2\pi] \times [0,a]$$

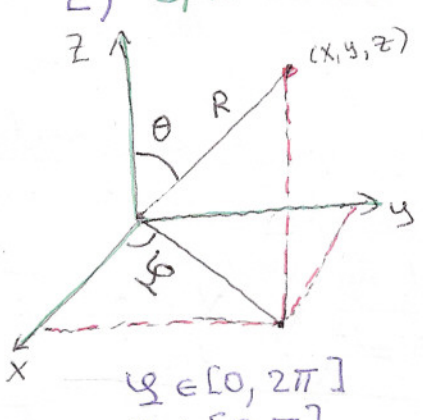
$$J = \begin{pmatrix} \cos\varphi & -R\sin\varphi & 0 \\ \sin\varphi & R\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det J = R$$

And

$$\iiint_{\Omega} f\,dx\,dy\,dz = \int_0^a \int_0^{2\pi} \int_0^b f\,dR\,d\varphi\,dz$$

2) Spherical coordinates – a bit more complicated



$\varphi \in [0, 2\pi]$
$\theta \in [0, \pi]$

$$x = R\sin\theta\cdot\cos\varphi$$
$$y = R\sin\theta\cdot\sin\varphi$$
$$z = R\cos\theta$$

old coordinates (x,y,z)
new (R,θ,φ)

Transformes: a ball $\Omega = \{ x^2+y^2+z^2 \le a^2 \}$ to the rectangular parallelepiped

$$D = [0,a] \times [0,\pi] \times [0,2\pi]$$

(useful for spherically symmetric problems, computations on a sphere, etc.)

Jacobi matrix:

$$J = \frac{\partial(x,y,z)}{\partial(R,\theta,\varphi)} = \begin{pmatrix} \sin\theta\cos\varphi & ; & R\cos\theta\cos\varphi & ; & -R\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & ; & R\cos\theta\sin\varphi & ; & R\sin\theta\cos\varphi \\ \cos\varphi & ; & -R\sin\theta & ; & 0 \end{pmatrix}$$

The Jacobian (determinant)

$$\det J = R^2 \sin\theta \quad - \text{ check!}$$

Finally

$$\iiint_\Omega f\,dx\,dy\,dz = \iiint_D R^2\sin\theta\, f\,dR\,d\theta\,d\varphi = \int_0^a \int_0^\pi \int_0^{2\pi} R^2\sin\theta \cdot f\,d\varphi\cdot d\theta\cdot dR$$

In particular, if f is spherically-symmetric

$$f(x,y,z) = f(\sqrt{x^2+y^2+z^2})$$

$$\iiint_\Omega f\,dx\,dy\,dz = \int_0^a \int_0^\pi \int_0^{2\pi} R^2 f(R)\cdot\sin\theta\,d\varphi\,d\theta\,dR = 2\pi \int_0^a R^2 f(R)\,dR \cdot \int_0^\pi \sin\theta\,d\theta = 4\pi \int_0^a R^2 f(R)\,dR$$

Volume of a Ball:

$$V = \iiint_\Omega 1\cdot dx\cdot dy\cdot dz = 4\pi \int_0^a R^2\,dR = \frac{4}{3}\pi a^3$$

PRACTICAL HINTS:

1. If the transform T is given always find first the new region $D = T^{-1}(\Omega)$
2. If not, try to find the transform which makes the domain rectangular (usually more important than simplifying the integrand)
3. Never (!) start computing antiderivatives before specifying the limits of integrations (in the iterated integrals).