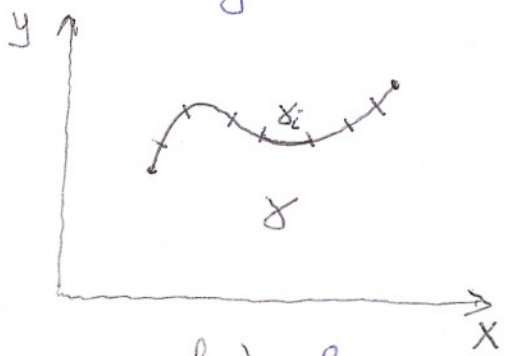
Lecture 20: Line and surface integrals
(scalar case)

Aims:
- to learn what are that integrals
- to derive the formulas for computing them

Start with line integrals (= curve integral = path integral = contour integral).

Def: (very similar to the usual integral) Let



γ be a (sufficiently smooth) curve in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and let $f: \gamma \to \mathbb{R}$ be a "good" function (continuous is enough). f can be defined on a curve only!

Consider a partition $\gamma = \bigcup_{i=1}^{n} \gamma_i$ on disjoint small pieces and take arbitrary points $s_i \in \gamma_i$. Then the integral sum for that partition is

$$I = \sum_{i=1}^{n} f(s_i)\,\delta l_i \qquad \delta l_i = \text{"length of } \gamma_i\text{"}$$

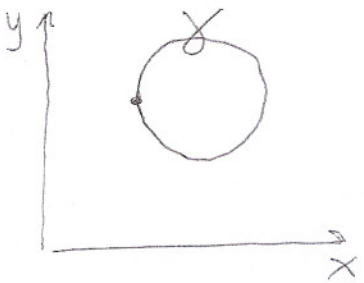
Finally, the integral is a limit of integral sums as the diameter of γ_i tends to zero.

$$\int_\gamma f\,dl = \lim_{\operatorname{diam}(\gamma_i)\to 0} \sum_{i=1}^{n} f(s_i)\,\delta l_i$$

As before, the limit exists for "good" functions f and "good" curves γ.

(NB): From def, the integral depends only on the curve γ and function f. The parametrization of γ is not involved in def $\Rightarrow$ The integral is independent of the way how we parametrize γ.

Notation: if γ is a loop, the following (closed curve, contour) notation is often used.



$$\oint_\gamma f\,dl$$

the circle indicates that the curve of integration is closed.

Next question: How to compute?

Answer: As before, by reducing to single integrals using some parametrization of γ.

Most general: The curve γ is given by

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}, \quad t \in [a,b] \qquad \text{OR} \qquad \begin{cases} x = x(t) \\ y = y(t) \end{cases} \text{ in 2D}$$

must be a one-to-one map between γ and $[a,b]$. Then, a partition of $[a,b]$ generates a partition of γ and we only need to understand what is the length of the piece of curve γ_i which corresponds to the interval $[t_i, t_i+\delta t]$.

$x(t)$, $y(t)$ and $z(t)$ are smooth = locally linear $\Rightarrow$

$$\delta x \approx x'(t_i)\delta t, \quad \delta y \approx y'(t_i)\delta t, \quad \delta z \approx z'(t_i)\delta t$$

and

$$\delta l_i \approx \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2 + [z'(t_i)]^2}\,\delta t$$

Thus

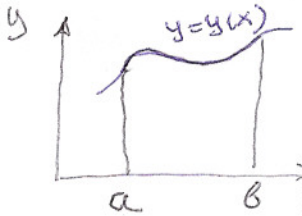
$$\oint_\gamma f\,dl = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

length differential

The desired f-la

(of course $z(t)$ is absent in 2D)

Example: 2D-case, γ is given by the graph $y = y(x)$, $x \in [a,b]$. Parametrization: $x = x$, $y = y(x)$



$x = t$ - parameter

$$\oint_\gamma f\,dl = \int_a^b f \cdot \sqrt{1+y'^2}\,dx$$

In partilar, $f \equiv 1$ gives the length of γ:

$$L = \int_a^b \sqrt{1+y'(x)^2}\,dx$$

- see Semester I.

Example: Length of a circle (circumference):

$$L = \oint_\gamma dl, \quad \gamma = \{x^2+y^2 = a^2\}$$

Parametrization:

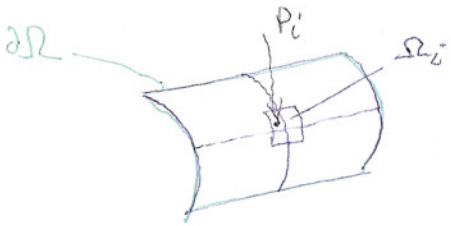
$$x = a\cos\varphi$$
$$y = a\sin\varphi$$

the simplest one.

$$L = \int_0^{2\pi} \sqrt{a^2\sin^2\varphi + a^2\cos^2\varphi}\, d\varphi = \int_0^{2\pi} a\, d\varphi = 2\pi a$$

(as expected!)

Surface integrals: We are now given a (non-flat) surface Ω in $\mathbb{R}^3$ + a "good" function $f: \Omega \to \mathbb{R}$



may be with boundary $\partial\Omega$ (like plotted) or without boundary, like a sphere

Def: Very close to the def of double integration: Consider a partition

$$\Omega = \bigcup_{i=1}^n \Omega_i$$ - disjoint union

+ $P_i \in \Omega_i$ $\quad \delta S_i := \text{AREA}(\Omega_i)$

The integral sums

$$I = \sum_{i=1}^n f(P_i)\,\delta S_i$$

And

$$\iint_\Omega f\, dS = \lim_{\operatorname{diam}(\Omega_i)\to 0} \sum_{i=1}^n f(P_i)\,\delta S_i$$

As before, the limit exists if f and Ω are "good" enough.

NB: Again, the def uses only the surface Ω, not it's parametrization $\Rightarrow$ integral is INDEPENDENT of the way how we parametrize the surface.

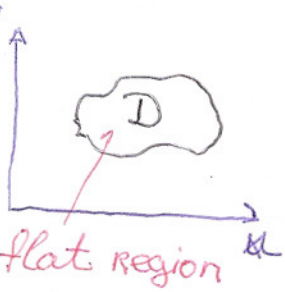
NEXT QUESTION: How to compute surface integrals?

Answer: By reducing to double integrals using the proper parametrization of Ω.

Most general:

$$\vec{R} = \vec{R}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$(u,v) \in D$



$\vec{R}: D \to \Omega$

must be one-to-one

Example: $\Omega = \{x^2+y^2+z^2 = a^2\}$ (sphere) can be parametrized by spherical coordinates

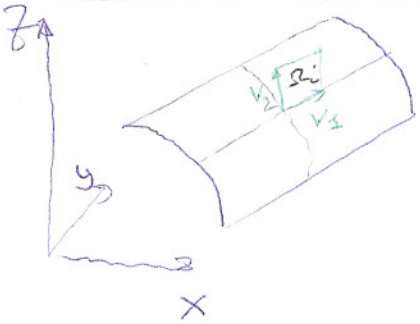
$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\cdot\sin\varphi \\ z = a\cos\theta \end{cases}$$

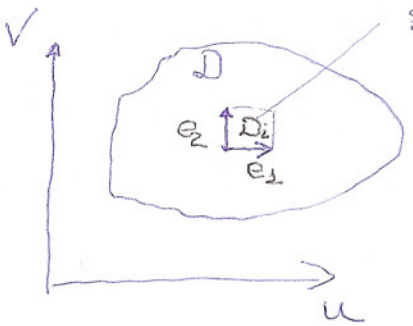
(θ, φ) – coordinates on a sphere

$$\begin{cases} 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{cases}$$

"almost" one-to-one

General formula:





small rectangle D_i generated by vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\delta u \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta v$$

Need to compute the AREA of $\Omega_i = \vec{R}(D_i)$ = the image of D_i on Ω.

D_i is small and $\vec{R}$ is small $\Rightarrow$ up to neglectable H.O.T. $\vec{R}$ is linear with the matrix given by Jacobi matrix

$$J = \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \\ z'_u & z'_v \end{pmatrix}$$

$\Rightarrow$ up to H.O.T. Ω_i is a parallelogram spanned by vectors

$$V_1 = J(e_1) = \begin{pmatrix} x'_u \\ y'_u \\ z'_u \end{pmatrix}\delta u = \vec{R}'_u \cdot \delta u$$

and

$$V_2 = J(e_2) = \begin{pmatrix} x'_v \\ y'_v \\ z'_v \end{pmatrix}\delta v = R'_v\,\delta v$$

V_1 and V_2 – pair of tangent vectors to Ω.

The area of that parallelogram is the modulus of the cross product

$$\delta S = |V_1 \times V_2| = |\vec{R}'_u \times \vec{R}'_v|\,\delta u\,\delta v$$

And the desired f-la:

$$\iint_\Omega f\,dS = \iint_D f\cdot|\vec{R}'_u \times \vec{R}'_v|\,du\,dv$$

curved area differential

NB: The normal vector $\vec{n}$ to the surface Ω is given by $\vec{n} = \vec{R}'_u \times \vec{R}'_v$ (will be used later).

Example: Ω is given by the graph $\boxed{z = f(x,y)}$

Then, the parametrization is

$$\boxed{\vec{R}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}} \qquad \boxed{\vec{R}'_x = \begin{pmatrix} 1 \\ 0 \\ f'_x \end{pmatrix}}, \quad \boxed{\vec{R}'_y = \begin{pmatrix} 0 \\ 1 \\ f'_y \end{pmatrix}}$$

$$\vec{R}'_x \times \vec{R}'_y = \begin{pmatrix} -f'_x \\ -f'_y \\ 1 \end{pmatrix} \quad \text{(check!)} \qquad \boxed{d\sigma = \sqrt{1 + f'^2_x + f'^2_y}}$$

$$\boxed{\iint\limits_{\Omega} f\, d\sigma = \iint\limits_{D} f \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, du\, dv}$$

Example: surface of revolution: obtained by Rotation of the curve $\boxed{\begin{array}{l} y = f(x) \\ a \le x \le b \end{array}}$ over the x-axis in 3D

failed to plot a picture!

natural parametrization:

$$\boxed{\vec{R}'_x = \begin{pmatrix} 1 \\ f'\cos\varphi \\ f'\sin\varphi \end{pmatrix}} \qquad \boxed{\vec{R}'_\varphi = \begin{pmatrix} 0 \\ -f\sin\varphi \\ f\cos\varphi \end{pmatrix}}$$

$$\boxed{\vec{R}(x,\varphi) = \begin{pmatrix} x \\ f(x)\cos\varphi \\ f(x)\sin\varphi \end{pmatrix}} \qquad 0 \le \varphi \le 2\pi$$

$$\vec{R}'_x \times \vec{R}'_\varphi = \begin{pmatrix} f'f \\ -f\cos\varphi \\ -f\sin\varphi \end{pmatrix} \quad \text{(check!)} \qquad \boxed{d\sigma = |f| \sqrt{1 + (f')^2}\, dx\, d\varphi}$$

$$\boxed{\iint\limits_{\Omega} f\, d\sigma = \iint\limits_{[a,b]\times[0,2\pi]} f \cdot |f| \sqrt{1 + (f')^2}\, dx\, d\varphi}$$

In particular, $f \equiv 1$ corresponds to the area of Ω:

$$\boxed{\sigma = \int_a^b \int_0^{2\pi} |f| \sqrt{1 + f'^2}\, dx\, d\varphi = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx}$$

(see semester 1)

Example: $\boxed{\Omega = \{x^2 + y^2 + z^2 = a^2\}}$ - sphere + spherical coord.

$$\boxed{\vec{R}(\theta,\varphi) = \begin{pmatrix} a\sin\theta\cos\varphi \\ a\sin\theta\sin\varphi \\ a\cos\theta \end{pmatrix}}$$

complete the calculations!

(see the last problem of a Coursework!)