Lecture 21 : Line and surface integrals (vector case)

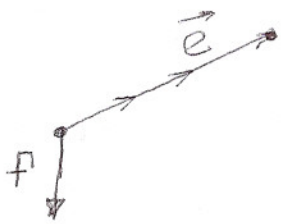
Aims:

- to understand why do we need to integrate vector fields
- learn how to find the proper tangent/normal vector field to a line/surface (orientation)
- to learn how to reduce the vector case to the scalar one.

I Motivation:

1) Example from mechanics : work of a force

$\vec{F} = (F_1, F_2, F_3)$ - force (vector) + straitlinear motion with displacement $\vec{\ell} = (\ell_1, \ell_2, \ell_3)$



Work $W = \vec{F} \cdot \vec{\ell} = F_1\ell_1 + F_2\ell_2 + F_3\ell_3$

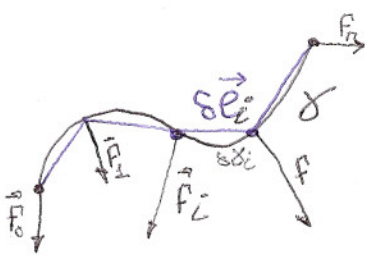
Reminder (see school or "classical dynamics")

Work = change of kinetic energy

(consequence of 2nd Newton law)

Now the motion is curvilinear and the force depends on the position: $\vec{F} = \vec{F}(x, y, z)$ - vector field

How to compute W in this case?



Answer: By localization, on a very small piece $\delta\gamma_i$ of the curve γ the motion is "almost" straightlinear and the force is "almost" constant

$$\Rightarrow \boxed{W_\gamma = \sum W_{\delta\gamma_i}}, \quad \boxed{W_{\delta\gamma_i} \approx \vec{F}_i \cdot \delta\vec{\ell}_i}, \quad \boxed{\delta\vec{\ell}_i \approx \vec{\tau}_i \cdot \delta\ell_i}$$

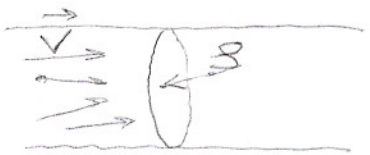
where $\vec{\tau}_i$ = "unit tangent vector" $\delta\ell_i$ = "length of γ_i"

$\Rightarrow$ in the limit $\boxed{W_\gamma = \int_\gamma \vec{F} \cdot \vec{\tau}\, d\ell}$

$\vec{\tau}$ = unit tangent vector to γ at $(x, y, z) \in \gamma$.

$\Rightarrow$ W is a line integral of a scalar function $\vec{F}\cdot\vec{\tau}$ = dot product of F with the unit tangent vector $\vec{\tau}$ to the curve γ.

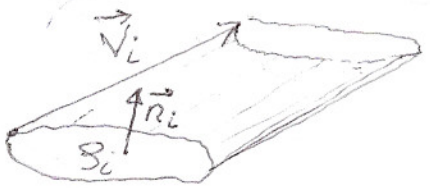
2) Example from fluid dynamics: mass flux through a given surface S.



Imagine: liquid moving in a tube with a cross-section S. At every point (x,y,z) we have the velocity $\vec{V}=\vec{V}(x,y,z)$ and density $\rho=\rho(x,y,z)$

Want to compute mass of liquid passed through S for the unit of time = mass flux through S

Again use the localization $\quad \text{Flux}(S)=\sum_i{}' \text{Flux}(S_i)$



$\vec{V}\approx\vec{V}_i$, $\rho\approx\rho_i$ – "almost" constants on a small piece S_i of S

$\Rightarrow \quad \text{Flux}(S_i)\cong \rho_i\cdot \text{Volume}_i$

Volume of the oblique cylinder $= \underbrace{S_i}_{\text{area of a base}}\cdot\underbrace{(\vec{V}_i\cdot\vec{n}_i)}_{\text{height}}$

$\vec{n}=\vec{n}(x,y,z)$ – unit normal to S at (x,y,z)

$$\Rightarrow \quad \text{Flux}=\iint_S \rho\vec{V}\cdot\vec{n}\,dS$$

Conclusion: if $\vec{F}$ – given vector field (smooth enough) we need to study

in 3D: Line integrals: $\int_\gamma \vec{F}\cdot\vec{\tau}\,dl$

Surface integrals: $\iint_S \vec{F}\cdot\vec{n}\,dS$

in 2D: Line integrals $\int_\gamma \vec{f}\cdot\vec{\tau}\,dl$ and $\int_\gamma \vec{f}\cdot\vec{n}\,dl$

To compute the above integrals, we need formulas for the tangent unit vector $\vec{\tau}$ to the curve γ and normal unit vector $\vec{n}$ to the surface S or curve γ in 2D.

Looks simple: indeed if the curve γ is parametrized by $x = x(t), y = y(t), z = z(t), t \in [a, b]$ $\Rightarrow$

tangent direction is $(x'(t), y'(t), z'(t))$ and

$$\vec{\tau} = \left(\frac{x'(t)}{\sqrt{(x')^2+(y')^2+(z')^2}}, \frac{y'(t)}{\sqrt{(x')^2+(y')^2+(z')^2}}, \frac{z'(t)}{\sqrt{(x')^2+(y')^2+(z')^2}} \right)$$

and if the surface S is parametrized by $\vec{R} = \vec{R}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$ then the normal direction is given by $\vec{R}'_u \times \vec{R}'_v$

and

$$\vec{n} = \frac{\vec{R}'_u \times \vec{R}'_v}{|\vec{R}'_u \times \vec{R}'_v|}$$

in 2D: if the curve γ is given by $x = x(t), y = y(t)$

then

$$\vec{\tau} = \left(\frac{x'(t)}{\sqrt{(x')^2+(y')^2}}, \frac{y'(t)}{\sqrt{(x')^2+(y')^2}} \right) \qquad \vec{n} = \left(\frac{y'(t)}{\sqrt{(x')^2+(y')^2}}, \frac{-x'(t)}{\sqrt{(x')^2+(y')^2}} \right)$$

since $\vec{\tau} \cdot \vec{n} \equiv 0$.

Be careful! if $\vec{\tau}$ / $-\vec{\tau}$ is a unit tangent vector than / is also and the same for $\vec{n}$!

$\Rightarrow$ there are 2 ways to fix the field of unit tangent vectors to γ in a continuous way: $\vec{\tau}$ and $-\vec{\tau}$.

The choice of this field is called orientation of γ.

Should be specified to compute the integral!

Indeed, $\int_\gamma \vec{F}(-\vec{\tau})\, d\ell = -\int_\gamma \vec{F} \cdot \vec{\tau}\, d\ell$ - depends on orientation!

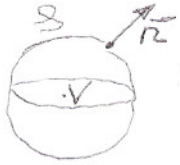
Analogously, for "good" surfaces S in $\mathbb{R}^3$ there are 2 possible ways to fix the unit normal: $\vec{n}$ and $-\vec{n}$

NB: There are non-orientable surfaces! Like Möbius strip

Non-interesting for integrals since all vector integrals are 0.

Important Rules for orientation:

3D: if S is a closed surface which restricts a Bounded volume V then the



outer normal is implicitly chosen (if the alternative choice is not explicitly chosen).

2D: if γ is a closed curve which restricts a Bounded area S then

$\vec{\tau}$ - corresponds to the anti-clockwise motion along the curve

$\vec{n}$ - is outer normal (if the alternative way of orintation is not explicitly fixed).

Computing vector integrals and alternative notations:

1) Line integrals: γ is parametrized by $\boxed{x = x(t),\ y = y(t),\ z = z(t)}$

Want to compute $\int_\gamma \vec{F}\cdot\vec{\tau}\,dl =$

$\boxed{dl = \sqrt{(x')^2+(y')^2+(z')^2}\,dt}$ - length differential

$\boxed{\vec{\tau} = \frac{1}{\sqrt{(x')^2+(y')^2+(z')^2}}(x', y', z')}$

$= \int_\gamma \left[f_1\cdot x'(t) + f_2\cdot y'(t) + f_3\cdot z'(t)\right]dt$ $\quad \sqrt{(x')^2+(y')^2+(z')^2}$ - cancels!

Remind: $dx = x'(t)dt$, $dy = y'(t)dt$, $dz = z'(t)dt$

$\Rightarrow$ very convenient notation

$$\boxed{\int_\gamma \vec{F}\cdot\vec{\tau}\,dl = \int_\gamma f_1\,dx + f_2\,dy + f_3\,dz}$$

to compute just replace dx, dy, dz by $x'(t)dt$, $y'(t)dt$, $z'(t)dt$ and Reduce it to usual integral w.r.t. t

2D case: Analogously:

$$\boxed{\int_\gamma \vec{F}\cdot\vec{\tau}\,dl = \int_\gamma f_1\,dx + f_2\,dy}$$

$$\boxed{\int_\gamma \vec{F}\cdot\vec{n}\,dl = \int_\gamma f_1\,dy - f_2\,dx}$$

if the orientantion of γ is anti-clockwise the automatically $\vec{n}$ is outer normal (check!)

Example: Let $\vec{F} = (x, y)$ and γ is a unit circle $x^2+y^2=1$

Need to compute: $\oint_\gamma \vec{F}\cdot\vec{\tau}\,dl = \oint_\gamma x\,dx + y\,dy$ and $\oint_\gamma \vec{F}\cdot\vec{n}\,dl = \oint_\gamma y\,dx - x\,dy$

Solution:

Parametrization: $x=\cos t$, $y=\sin t$, $t\in[0,2\pi]$ – anti-clockwise

$$\oint_\gamma x\,dx+y\,dy = \int_0^{2\pi}[\cos t\cdot(-\sin t)+\sin t\cdot\cos t]\,dt = 0$$

$$\oint_\gamma -y\,dx+x\,dy = \int_0^{2\pi}[-\sin t\cdot(-\sin t)+\cos t\cdot\cos t]\,dt = +\int_0^{2\pi} 1\,dt = +2\pi$$

2) **Surface integrals:** S is parametrized by $(u,v)\in D$

$$\vec{R} = \vec{R}(u,v) = \begin{pmatrix} x(u,v)\\ y(u,v)\\ z(u,v)\end{pmatrix},\quad \vec{n} = \frac{\vec{R}'_u\times\vec{R}'_v}{|\vec{R}'_u\times\vec{R}'_v|}$$

$dS = |\vec{R}'_u\times\vec{R}'_v|\,du\,dv$ – area differential

$\vec{F} = (f_1, f_2, f_3)$ – vector field $\Rightarrow$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot(\vec{R}'_u\times\vec{R}'_v)\,du\,dv = \iint_D \langle \vec{F}, \vec{R}'_u, \vec{R}'_v\rangle\,du\,dv$$

triple product

$$= \iint_D \det\begin{pmatrix} F_1, F_2, F_3\\ x'_u, y'_u, z'_u\\ x'_v, y'_v, z'_v\end{pmatrix} du\,dv = \iint_D \left(F_1\begin{vmatrix} y'_u, z'_u\\ y'_v, z'_v\end{vmatrix} + F_2\begin{vmatrix} x'_u, z'_u\\ x'_v, z'_v\end{vmatrix} + F_3\begin{vmatrix} x'_u & y'_u\\ x'_v & y'_v\end{vmatrix}\right) du\,dv$$

This justifies the following alternative notation

$$\iint_S \vec{F}\cdot\vec{n}\,dS \equiv \iint_S F_1\,dy\,dz - F_2\,dx\,dz + F_3\,dx\,dy$$

To compute → need to replace

$$dy\,dz \to \det\begin{pmatrix} y'_u & y'_v\\ z'_u & z'_v\end{pmatrix} = y'_u\cdot z'_v - y'_v\cdot z'_u$$

$$dx\,dz \to x'_u\cdot z'_v - z'_u\cdot x'_v$$

$$dx\,dy \to x'_u\cdot y'_v - y'_u\cdot x'_v$$

More advanced notation – using wedge product:

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S f_1\,dx\wedge dy + f_2\,dz\wedge dx + f_3\,dx\wedge dy$$

$dz\wedge dx = -dx\wedge dz$, Roughly speaking $\wedge$ means here the above 2×2 determinants.