Lecture 21: The analogues of Newton-Leibnitz formula for multiple integrals

Aims:

- to learn what are $\operatorname{div}\vec{F}$ and $\operatorname{curl}\vec{F}$
- to state Green, Gauss-Ostrogradsky and Stokes theorem

Reminder: Newton-Leibnitz formula = fundamental Th. of calculus: if f is continuous on $[a,b]$ and F is antiderivative of f ($F'(x)=f(x)$) $\Rightarrow$

$$\boxed{\int_a^b f(x)\,dx = F(b)-F(a)} \quad \text{OR} \quad \boxed{\int_a^b F'(x)\,dx = F(b)-F(a)}$$

The integral of F' over $[a,b]$ is expressed trough the values of F on the boundary $\boxed{\partial[a,b]=\{a,b\}}$

NB! Continuity of f is important:

$$\boxed{\int_{-1}^{1}\frac{dx}{x^2} \overset{?}{=} \frac{-1}{x}\Big|_{-1}^{1} = -1-1=-2} \;!?$$

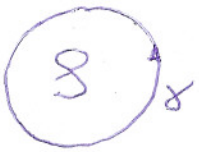
completely wrong answer! integral of positive function is negative !!!

Why? $f(x)=\frac{1}{x^2}$ is discontinuous at $x=0$ and the integral does not exist!

Want to find something similar for the multiple integrals.

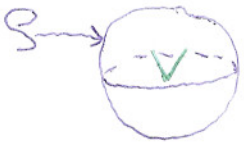
3 natural cases:

1) 2D an area S is restricted by a curve γ



Want to express the integral over S of the "derivative" of f in terms of the integral of f over the boundary curve γ.

2) 3D a volume V is restricted by a surface S



Want to express the integral over V through the surface integral over the boundary S

3) a surface S in 3D is restricted by the curve γ (boundary)

Want to express surface integral over S through curve integral over γ.

Answer: such formulas indeed exist (for all 3 cases), but to state them, we need to introduce two differential operators: div and curl

Def 1: Let $\vec{F} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ be smooth vector field. Then the divergence f is defined by

$$\boxed{\operatorname{div} \vec{F} := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}}$$

Alternative notation:

$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$ (do not mix with the gradient ∇f of a scalar function) = "dot product" of the gradient $\boxed{\nabla = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}}$ and a vector $f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$

in 3D $(x_1, x_2, x_3) = (x, y, z)$

$$\boxed{\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

div is defined analogously in any space dimension!

e.g. in 2D $\boxed{\operatorname{div} \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}}$

Example: $\boxed{\vec{F} = \begin{pmatrix} xy \\ xz \\ yz \end{pmatrix}}$ $\boxed{\operatorname{div} \vec{F} = \frac{\partial (xy)}{\partial x} + \frac{\partial (xz)}{\partial y} + \frac{\partial (yz)}{\partial z} = y + y = 2y}$

Def 2: Let $\vec{F}$ be smooth vector field. The curl (ROTOR) curl $\vec{F}$ is given by

$$\boxed{\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{pmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) e_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) e_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) e_3}$$

$\Rightarrow$ curl $\vec{F}$ is a vector field

$$\boxed{\operatorname{curl} \vec{F} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}}$$

= cross product of ∇ and $\vec{F}$.

(often also denoted by rot $\vec{F}$.

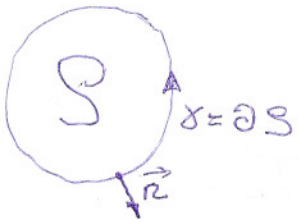
Example: $\vec{F} = \begin{pmatrix} xy \\ xz \\ yz \end{pmatrix}$ $\quad \operatorname{curl} \vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & yz \end{vmatrix} =$

$$= (z-x)e_1 + 0 \cdot e_2 + (z-x)e_3 \Rightarrow \operatorname{curl} \vec{F} = \begin{pmatrix} z-x \\ 0 \\ z-x \end{pmatrix}$$

We are now ready to state the desired theorems:

Case I: 2D Region S restricted by the curve γ

= GREEN THEOREM:



Let $F = (F_1, F_2)$ be smooth vector field in D (first derivatives must be continuous)

γ is a "good" curve $\Rightarrow$

$$\boxed{\oint_\gamma \vec{F} \cdot \vec{n}\, dl = \iint_S \operatorname{div} F\, dS}$$

Orientation:

$\vec{n}$ is outer normal = γ is passed anti-clock wise

Equivalent formulation:

$$\boxed{\oint_\gamma F_1\, dy - F_2\, dx = \iint_S \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dS}$$

see previous Lecture

OR denoting $f_1 = -F_2$, $f_2 = F_1 \Rightarrow$

$$\boxed{\oint_\gamma f_1\, dx + f_2\, dy = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dS}$$

NB: Any of that 3 formulas will be counted as a correct answer on "what is GREEN Th.". Do not forget orientation and smoothness of F (will be counted as a mistake!)

Proof: WRITE The double integral as iterated one and cancel one derivative with one integration using the N-L formula (NO DETAILS is given here!)

Example: Formula for the area of S through curve integral:

Let $\boxed{F = \frac{1}{2}(x, y)} \Rightarrow \boxed{\operatorname{div} \vec{F} = 1}$

$$\frac{1}{2} \oint_\gamma x\, dy - y\, dx = \iint_S 1\, dS = \text{"area of } S\text{"}$$

Case II: 3D Region V is restricted by the surface S — **GAUSS-Ostrogradsky Theorem**:

Let $\vec{F} = (F_1, F_2, F_3)$ be smooth vector field in V and S is a "good" surface $\Rightarrow$

$$\boxed{\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_V \operatorname{div}\vec{F}\, dV}$$

Orientation: $\vec{n}$ is outer normal

Very similar f-la to the 2D case (div Theorem) (valid in any space dimension!) and with very similar proof!

Case III: The surface S in 3D is restricted by the boundary γ — **Stokes Theorem**:

Let $\vec{F} = (F_1, F_2, F_3)$ be smooth vector field on S and S is a "good" surface with "good" boundary $\Rightarrow$

$$\boxed{\oint_\gamma \vec{F}\cdot\vec{\tau}\, dl = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS}$$

Orientation: A bit more delicate:
- S is not closed $\Rightarrow$ inner/outer normals are NOT Defined
  $\Rightarrow$ we just have 2 choices of $\vec{n}$
- anti-clockwise orientation is not defined for the curve in 3D (!)

Answer: $\vec{n}$ and $\vec{\tau}$ should be chosen using the Right-hand Rule:

if the thumb of a right hand points along $\vec{n}$, the fingers should curl in the direction of $\vec{\tau}$.

Remark: ALL these theorems are particular cases of a general Stokes Theorem (will be learned on "Differential geometry" at year 3).