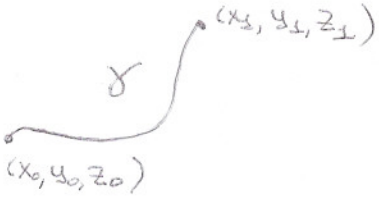
Lecture 23 (Last): Potential vector fields and potentials

Aims:
- learn what are the potential (conservative) vector fields
- understand how to recognize potential fields

Motivation: Let $\vec{F} = \vec{F}(x, y, z)$ be continuous vector field representing forces. Then



$$W = \int_\gamma \vec{F} \cdot \vec{\tau}\, dl = \int_\gamma F_1 dx + F_2 dy + F_3 dz$$

work done by forces $\vec{F}$.

From the Newtonian law:

$$E_k\big|_{(x_1,y_1,z_1)} - E_k\big|_{(x_0,y_0,z_0)} = W$$

change of the kinetic energy = work of all forces

see Classical dynamics

Assume: W is independent of the path γ and depends only on (x_0, y_0, z_0) and (x_1, y_1, z_1). Then, as we will see

(*) $$W = \varphi(x_0, y_0, z_0) - \varphi(x_1, y_1, z_1)$$, φ - a scalar function

φ is called potential of $\vec{F}$.

Then

$$E := E_K + \varphi\big|_{(x_0,y_0,z_0)} = E_K + \varphi\big|_{(x_1,y_1,z_1)}$$

E: total energy; E_K: kinetic energy; φ: potential energy

(*) implies the energy concervation law (!)

Important: What forces are potential (conservative)? How to find the potential (energy)?

Step 1: If $\vec{F}$ is potential. Then

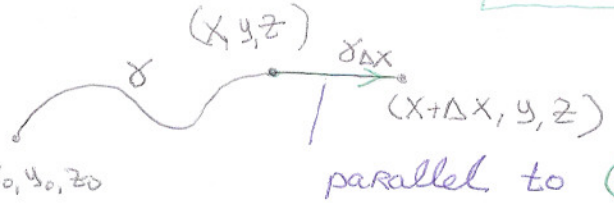
$$\boxed{\vec{F} = -\nabla \mathcal{P}} \quad \text{for some smooth } \mathcal{P}.$$

Proof: Fix (x_0, y_0, z_0) and set

$$\boxed{\mathcal{P}(x,y,z) := -\int_\gamma \vec{F}\cdot\vec{\tau}\, d\ell}$$

γ is any path connecting (x_0, y_0, z_0) and (x, y, z)

Let us prove that $\boxed{\frac{\partial}{\partial x}\mathcal{P} = -F_1}$ other derivatives are analogous.



parallel to (x)-axis

$$\mathcal{P}(x+\Delta x, y, z) - \mathcal{P}(x,y,z) =$$
$$= -\int_{\gamma \cup \gamma_{\Delta x}} \vec{F}\cdot\vec{\tau}\, d\ell + \int_\gamma \vec{F}\cdot\vec{\tau}\, d\ell =$$
$$= -\int_{\gamma_{\Delta x}} \vec{F}\cdot\vec{\tau}\, d\ell = -\int_x^{x+\Delta x} F_1\, dx$$

$$\Rightarrow \boxed{\frac{\mathcal{P}(x+\Delta x, y, z) - \mathcal{P}(x,y,z)}{\Delta x} = -\frac{1}{\Delta x}\int_x^{x+\Delta x} F_1\, dx}$$

$$\Delta x \to 0 \Rightarrow \boxed{\frac{\partial \mathcal{P}}{\partial x} = -F_1}$$

Step 2: If $\boxed{\vec{F} = \nabla \mathcal{P}}$ then (*) holds

(the integral is path-independent)

Proof: Let $\boxed{(x(t), y(t), z(t))}$ be a smooth path connecting $(x_0, y_0, z_0)\,|_{t=0}$ and $(x_1, y_1, z_1)\,|_{t=1}$

$$W = -\int_\gamma \frac{\partial \mathcal{P}}{\partial x}dx + \frac{\partial \mathcal{P}}{\partial y}dy + \frac{\partial \mathcal{P}}{\partial z}dz = -\int_0^1 \left[\frac{\partial \mathcal{P}}{\partial x}x'(t) + \frac{\partial \mathcal{P}}{\partial y}y'(t) + \frac{\partial \mathcal{P}}{\partial z}z'(t)\right]dt =$$
$$= -\int_0^1 \frac{d}{dt}\mathcal{P}(x(t), y(t), z(t))\,dt = \mathcal{P}(x_0, y_0, z_0) - \mathcal{P}(x_1, y_1, z_1)$$

independent of the path

$$\Rightarrow \boxed{\text{Potential (conservative) vector fields}} = \boxed{\text{Gradient vector fields}}$$

Problem: Still do not know how to recognize gradient fields

$$\boxed{\vec{F} = -\nabla \mathcal{P}}\ ?$$

Step 3: $\boxed{\vec{F} = -\nabla \varphi} \Rightarrow \boxed{\text{curl } \vec{F} = 0}$

φ is smooth

Proof: $f_1 = -\varphi'_x$, $f_2 = -\varphi'_y$, $f_3 = -\varphi'_z$

$$\left.\begin{aligned} \partial_y f_1 - \partial_x f_2 &= -\varphi''_{xy} + \varphi''_{yx} = 0 \\ \partial_z f_1 - \partial_x f_3 &= -\varphi''_{xz} + \varphi''_{zx} = 0 \\ \partial_z f_2 - \partial_y f_3 &= -\varphi''_{yz} + \varphi''_{zy} = 0 \end{aligned}\right| \Rightarrow \boxed{\text{curl } \vec{F} = 0}$$

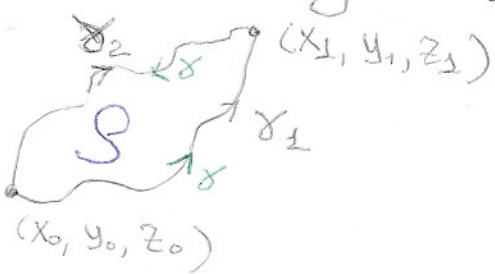
easy to check!

Step 4 (final): Let $\vec{F}$ be smooth vector field in the whole space (or in a simply connected domain D) such that

$\boxed{\text{curl } \vec{F} \equiv 0}$

$\Rightarrow \vec{F}$ is potential. $\Rightarrow \boxed{\vec{F} = -\nabla \varphi}$

Proof: We need to show that W is path-independent. Let γ_1 and γ_2 be 2 paths connecting (x_0, y_0, z_0) and (x_1, y_1, z_1)



need to prove

$$\boxed{\int_{\gamma_1} \vec{F} \cdot \vec{\tau}\, dl = \int_{\gamma_2} \vec{F} \cdot \vec{\tau}\, dl}$$

Let γ be a loop $\gamma_1 \cup \gamma_2$ (with opposite orientation)

$\Rightarrow$ need to check $\boxed{\oint_\gamma \vec{F} \cdot \vec{\tau}\, dl = 0}$

By Stokes theorem

Let S be a smooth surface with boundary γ (always exists in $\mathbb{R}^3$ if γ_1 and γ_2 do not intersect — used without proof!!!)

$$\Rightarrow \boxed{\oint_\gamma \vec{F} \cdot \vec{\tau}\, dl = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS = 0}$$

"0"

Remark: 2D $\vec{F} = (f_1, f_2)$

Necessary and sufficient condition:

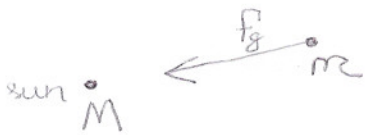
$$\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} \equiv 0$$

only 1 equation instead of 3 in 3D.

Example: 1) Gravitation: $\vec{F} = m\vec{g}$

Potential $\Phi = +mgz$ corresponds to "school"

2) Newton gravitation law: $\Pi = mgh$ - potential energy



$$\vec{f}_g = -G\frac{Mm}{R^2}\cdot\frac{\vec{R}}{R}$$

m, M - masses G - gravitational const.

also potential with

$$\Phi(x,y,z) = -\frac{GMm}{R}, \quad R = (x^2+y^2+z^2)^{1/2}$$

check!

Example: Let

$$f_2 = \frac{x}{x^2+y^2}, \quad f_1 = \frac{-y}{x^2+y^2}$$

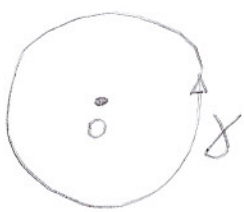
This vector field is smooth everywhere except of the origin and

$$\partial_y f_1 - \partial_x f_2 \equiv 0$$ - check!

However:

$$\oint_\gamma \frac{x\,dy}{x^2+y^2} - \frac{y\,dx}{x^2+y^2} \neq 0$$

check that it is 2π if γ is a circle $x^2+y^2=1$



Reason: Green formula does not work due to the singularity at the origin!

$\Rightarrow$ Residual theory, see Complex analysis