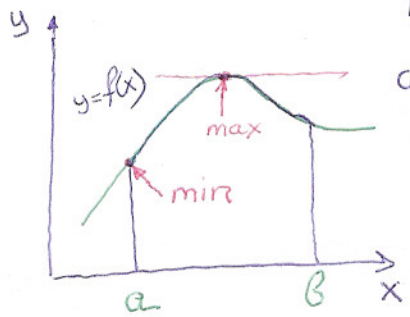
Lecture 7-8: Max and min for functions of many variables with constrains.

Aim: - to learn how to find max / min of a function in a region
- to understand the Lagrange multipliers method.

1D Reminder:



Problem: To find the maximal and minimal value of a smooth function on a segment $D = [a, b]$

Solution: 1) Any continuous function on a segment D attains it max and min value (important theorem from Real analysis).

$\Rightarrow$

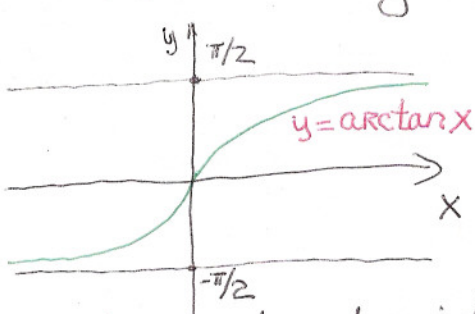
2) Max / min is attained either inside of D (at $x_0 = (a, b)$)
$\Rightarrow$ x_0 is a stationary point OR on the boundary
($f'(x_0) = 0$) $x_0 = a$ or $x_0 = b$

3) **Algorithm:** a) to find all stationary points of f = solutions of $f'(x_0) = 0$ on (a, b)

b) Compute the values of f at all stationary points + values at a and b

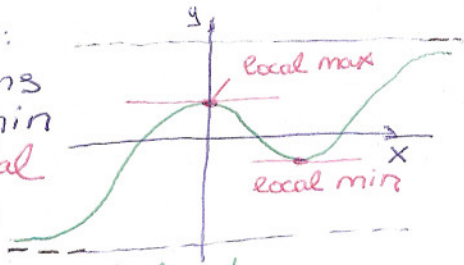
c) minimal of that values will be global minimum of f in D and the maximum of them = global maximum of f in D.

More complicated if D is unbounded, say, $D = \mathbb{R}_+$ or $D = \mathbb{R}$
max / min may not exist!!



max and min do not exist!
no stationary points
only sup and inf exist.

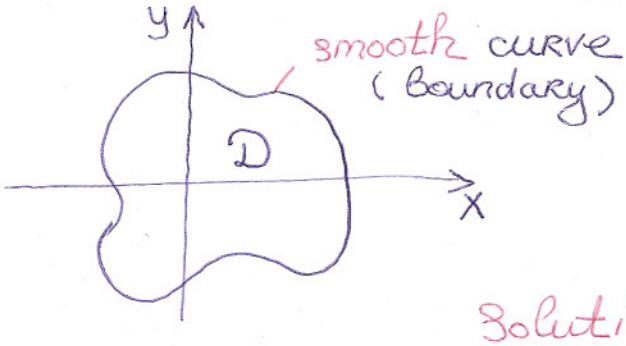
one more example:
- 2 stationary points
- local max and min
- Nevertheless, global max and min do NOT exist!!!



$\Rightarrow$ easy to make a mistake!
Extra attention is required when D is unbounded!

Many variables (start with many = 2)

$$f = f(x, y)$$



Problem: find minimal and maximal value of a smooth function f in a region D.

Solution: 1) Again Th. from Real analysis: Any continuous function in a bounded (Boundary curve is included) region D attains it's max and min value.

2) $\Rightarrow$ max/min is either inside of D (stationary point) $\nabla f = 0$ OR on the Boundary!

Main difference: The Boundary now consists of infinitely many points! (smooth curve in our situation).

Thus, we have already learned how to find stationary points inside of D (need to solve $\nabla f = 0$ = find all solutions of $\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases}$ belonging to D)

But we still do not know how to find "stationary" points on the Boundary curve!

necessary to learn how to do that in order to solve max/min problem in a region D.

Example: find min/max of $f = x^2 - y^2$ in a disk $x^2 + y^2 \le 1$

1) Stationary points inside of D: $\nabla f = (2x, -2y) = 0 \Rightarrow (x, y) = (0, 0)$

$\Rightarrow$ one stationary point $\Rightarrow$ saddle

$\Rightarrow$ Both max and min are attained on the Boundary circle $x^2 + y^2 = 1$

$\Rightarrow$ need to find max / min of $\boxed{f = x^2 - y^2}$

on a circle $\boxed{x^2 + y^2 = 1}$ ← constrain on the admissible (x, y).

NEW TYPE of PROBLEMS!

In this simple case there is a straightforward way to solve it.

Indeed, let us parametrize a circle by $\boxed{\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}}$

$\Rightarrow$ need to find min/max of $\boxed{x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t}$

No more calculus is needed to see that

$\boxed{\max = +1}$, $\boxed{\min = -1}$

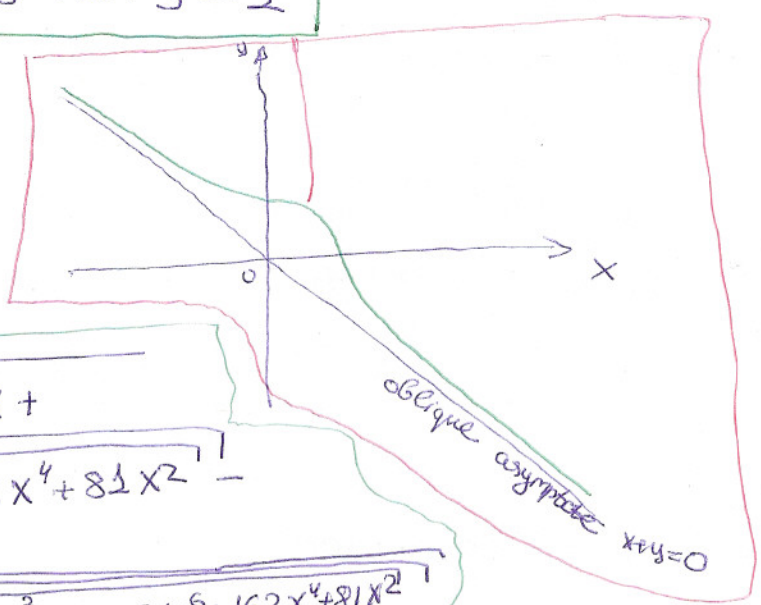
Why is that method "Bad" in general?

It can be extremely difficult to parametrize the constrain curve or to solve it explicitly with respect to x, or y.

"Terrifying" example: Let the constrain be a cubic curve:

$$\boxed{x^3 + y^3 + x + y = 1}$$

$\Rightarrow$ Relatively simple algebraic curve which looks like



But an attempt to write it as $\boxed{y = y(x)}$ gives

$$y(x) = \frac{1}{6}\sqrt[3]{108 - 108x^3 - 108x + 12\sqrt{93 - 162x^3 - 162x + 81x^6 + 162x^4 + 81x^2}} - \frac{2}{\sqrt[3]{108 - 108x^3 - 108x + 12\sqrt{93 - 162x^3 - 162x + 81x^6 + 162x^4 + 81x^2}}}$$

found by Cardano's formula via Maple

$\Rightarrow$ Need to be able to find stationary points without solving the constrain equation !!!

Lagrange (multipliers) method:

Problem: find min/max $z = f(x,y)$ under the constrain $g(x,y)=0$ — implicit equation for a curve

So, alternatively, find min/max $z = f(x,y)$ on a curve $g(x,y)=0$

e.g. on a circle $g = x^2+y^2-1$ as in the last example

1 st order test (stationary points) is needed.

As before, we use Linear approximations:

$$\delta z = f'_x \delta x + f'_y \delta y + \text{H.O.T.}$$

f'_x and f'_y are computed at (x_0, y_0) Belonging to the curve $g(x_0,y_0)=0$

$\delta z := z - z_0$
$\delta x := x - x_0$
$\delta y := y - y_0$

But now the increments δx and δy are not arbitrary, but constrained by

$$0 = \delta g = g'_x \delta x + g'_y \delta y + \text{H.O.T.}$$

since only (x,y) Belonging to the curve are allowed ($g(x,y)=0$)

Neglecting the H.O.T., we see that δz can be sign-def only if

$$f'_x\, dx + f'_y\, dy = 0$$

for all (dx, dy) satisfying

$$g'_x\, dx + g'_y\, dy = 0$$

In other words, for every $\begin{pmatrix} dx \\ dy \end{pmatrix} \perp \nabla g$

we must have $\begin{pmatrix} dx \\ dy \end{pmatrix} \perp \nabla f$

That is possible only if $\nabla f \parallel \nabla g$

two vectors ∇f at ∇g computed at (x_0, y_0) must be parallel

The desired 1st order test!

Thus (x_0, y_0) is a stationary point for $\boxed{z = f(x,y)}$ on a curve $\boxed{g(x,y)=0}$

if $g(x_0,y_0)=0$ AND

(*) $\boxed{\nabla f|_{(x_0,y_0)} \parallel \nabla g|_{(x_0,y_0)}}$

the only candidates for min/max in a smooth case!

If $\nabla g = 0$ (*) is always satisfied $\Rightarrow$ stationary points

Let $\boxed{\nabla g \neq 0}$ Then (*) $\Rightarrow$

$$\boxed{\nabla f = \lambda \nabla g}$$ for some λ to be found.

So, we have 3 equations for the triple (x, y, λ)

(*,*)
$$\begin{cases} g(x,y)=0 \text{ - constrain} \\ f'_x(x,y) + \lambda g'_x(x,y) = 0 \\ f'_y(x,y) + \lambda g'_y(x,y) = 0 \end{cases}$$

parallelity condition

Convenient notations:

Let $\boxed{L_\lambda(x,y) := f(x,y) + \lambda g(x,y)}$, $\lambda \in \mathbb{R}$

be the Lagrange function associated with our problem. It depends on the parameter λ called Langrange multiplier.

Then the last two equations read

$$\boxed{\nabla L_\lambda(x,y) = 0}$$

equation for the usual stationary points, without constrains, but the value of λ should be determined by $\boxed{g(x,y)=0}$

Example: Find min/max of $\boxed{z = x+y}$ in the region $\boxed{2x^2+y^2 \leq 1}$ interior of the ellipse.

Step 1: Stationary points inside:

$\nabla z = (1,1) = 0$ - impossible $\Rightarrow$ min and max "live" on the boundary curve $\boxed{2x^2+y^2=1}$

Need to solve the constrained problem

$z = x+y \to \max$ or $2x^2+y^2-1=0$

Step 2: Lagrange function

$$L_\lambda(x,y) = x+y+\lambda(2x^2+y^2-1)$$

$$\nabla L_\lambda = (1+4\lambda x,\ 1+2\lambda y) = 0$$

$x = -1/4\lambda$, $y = -\frac{1}{2\lambda}$ need to be inserted to the constrain equation

$2\frac{1}{16\lambda^2} + \frac{1}{4\lambda^2} = 1$ $\quad 16\lambda^2 = 6$ $\quad \lambda = \pm\sqrt{6}/4$

$\Rightarrow$ two stationary points

$(x_0, y_0) = (-1/\sqrt{6}, -\frac{2}{\sqrt{6}})$, $z_0 = -3/\sqrt{6}$

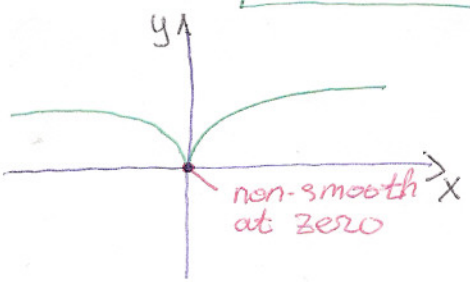
$(x_1, y_1) = (1/\sqrt{6}, 2/\sqrt{6})$, $z_1 = 3/\sqrt{6}$

Step 3: Need to add points where $\nabla g = (4x, 2y) = 0$

$\Rightarrow (x,y) = 0$ – does not belong to the curve.

At such points, the curve $g(x,y)=0$ may be not smooth

example $y^3 - x^2 = 0$ (All such points should be in the list!)



Final answer: $\max = +3/\sqrt{6}$, $\min = -3/\sqrt{6}$

Extensions: 1) more than 2 variables $\to$ immediate

Nothing changes except the gradient which will have more than 2 components!

2) More constrains: e.g. 3 variables + 2 constrains

$f = f(x,y,z) \to \max/\min$ under $g(x,y,z) = h(x,y,z) = 0$

Rule: every constrain needs its own Lagrange multiplier

Number of constrains = Number of Lagrange multipliers

In that case

$$L_\lambda(x,y,z) = f + \lambda_1 g + \lambda_2 h$$

5 eqn. on 5 unknowns $(x,y,z,\lambda_1,\lambda_2)$

$\nabla L_\lambda(x,y,z) = 0$ – 3 eqns.

$+\ g(x,y,z) = 0$

$+\ h(x,y,z) = 0$.