Lecture 9. Vector functions of many variables.

Aim: - To learn how to compute differentials of vector functions.

- To learn what is Jacobian and how the inverse function theorem can be extended to many variables.

Reminder: scalar functions of many variables

$$f: \mathbb{R}^n \to \mathbb{R}, \quad x \to f(x)$$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ - vector independent variable → scalar dependent variable

$$y = f(x_1, \dots x_n)$$

f

In that case

$$\delta f = f'_{x_1}\delta x_1 + \dots + f'_{x_n}\delta x_n + \text{H.O.T.} \Rightarrow df = \nabla f \cdot dx$$

$$\delta f = f(x) - f(x^0), \quad \delta x_i = x_i - x_i^0$$

$\Rightarrow$ Total derivative is a vector (called gradient)

$$\nabla f = (f'_{x_1}, \dots, f'_{x_n})$$

Now, look at the general case

$$f: \mathbb{R}^n \to \mathbb{R}^m \qquad y = f(x)$$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ - vector independent variable (n-vector) → $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ - vector dependent variable (m-vector)

in coordinates

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = f(x) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots x_n) \end{pmatrix}$$

= m - scalar functions

Natural examples:

a) Vector fields. For instance, velocity field in a fluid

b) Change of coordinates. For instance

polar coordinates $\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$ is an $\mathbb{R}^2 \to \mathbb{R}^2$ map $\begin{pmatrix} R \\ \varphi \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$

How to compute differentials?

$f: \mathbb{R}^n \to \mathbb{R}^m$, $y = f(x)$

Now, we have n-vector of increments $\delta x = \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix}$ of independent variable x and m-vector of increments $\delta y = \begin{pmatrix} \delta y_1 \\ \vdots \\ \delta y_m \end{pmatrix}$ of dependent variables.

Solution: We may compute every $\delta y_i = \delta f_i$ separately as increments of scalar functions.

$$\begin{cases} \delta y_1 = \delta f_1 = f'_{1,x_1}\delta x_1 + \cdots + f'_{1,x_n}\delta x_n + H.O.T. \\ \vdots \\ \delta y_m = \delta f_m = f'_{m,x_1}\delta x_1 + \cdots + f'_{m,x_n}\delta x_n + H.O.T. \end{cases}$$

OR in vector notations

$$\delta y = \begin{pmatrix} \delta y_1 \\ \vdots \\ \delta y_m \end{pmatrix} = \begin{pmatrix} f'_{1,x_1}, \cdots, f'_{1,x_n} \\ \vdots \\ f'_{m,x_1}, \cdots f'_{m,x_n} \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{pmatrix} + H.O.T.$$

$m \times n$-matrix of partial derivatives

This matrix has a special name – Jacobi matrix of the map f:

!! important notion

$$J(f) = J_{(x)}(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, \cdots, \frac{\partial f_1}{\partial x_n} \\ \vdots \\ \frac{\partial f_m}{\partial x_1}, \cdots, \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

(x) used to indicate the independent variable

Thus, neglecting the higher order terms

$$dy = J_x(f)\,dx$$

$\Rightarrow$ Total derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$ is an $m \times n$-matrix – Jacobi matrix $J(f)$

One more useful notation (we will not use it, but can be easily found in some textbooks!)

$$\frac{\partial(f_1, \cdots f_m)}{\partial(x_1, \cdots x_n)} = J(f)$$

$\Rightarrow$ Linearization $y = f(x)$ is (near $x^0 \in \mathbb{R}^n$, $y^0 = f(x^0) \in \mathbb{R}^m$)

$$y - y^0 = J(f)\big|_{x=x^0} \cdot (x - x^0)$$

= equation for the "tangent space" of this map at $x = x^0$.

Examples:

1) Scalar function $f: \mathbb{R}^n \to \mathbb{R}$, $m = 1$

Jacobian is $1 \times n$ matrix = (co) vector = gradient

$$J(f) = \nabla f = (f'_{x_1}, \dots, f'_{x_n})$$

(Consistency with the previous theory!)

2) Vector function of scalar variable $f: \mathbb{R} \to \mathbb{R}^m$, $n = 1$

$y = f(x) = \begin{pmatrix} f_1(x) \\ f_m(x) \end{pmatrix}$ - For instance, trajectory of a particle depending on time x.

Jacobi matrix is $m \times 1$ - matrix = m-vector

$$J(f) = \begin{pmatrix} f'_1(x) \\ f'_m(x) \end{pmatrix}$$

again, in a complete agreement with the previous knowledge.

Most interesting for us are changes of variables:

3) Polar coordinates in $\mathbb{R}^2$: $f: \begin{pmatrix} R \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} \qquad \begin{cases} dx = dR\cdot\cos\varphi - R\sin\varphi\cdot d\varphi \\ dy = \sin\varphi\, dR + R\cos\varphi\, d\varphi \end{cases}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\varphi, & -R\sin\varphi \\ \sin\varphi, & R\cos\varphi \end{pmatrix} \begin{pmatrix} dR \\ d\varphi \end{pmatrix}$$

and

$$J = \frac{\partial(x,y)}{\partial(R,\varphi)} = \begin{pmatrix} \cos\varphi, & -R\sin\varphi \\ \sin\varphi, & R\cos\varphi \end{pmatrix}$$

4) One more change of variables on a plane:

$$\begin{cases} x = u + v \\ y = u - v \end{cases}$$

Linear map $\Rightarrow$ Linearization of a linear map = the same map:

$$\begin{cases} dx = du + dv \\ dy = du - dv \end{cases} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 1, & 1 \\ 1, & -1 \end{pmatrix}$$

5) General change of variables on a plane:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$x(u,v)$ and $y(u,v)$ - some smooth functions.

$$\begin{cases} dx = x'_u\, du + x'_v\, dv \\ dy = y'_u\, du + y'_v\, dv \end{cases} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} x'_u, & x'_v \\ y'_u, & y'_v \end{pmatrix}$$

Inverse maps: Let $f: \mathbb{R}^n \to \mathbb{R}^n$ (important that $m=n$!)

$y = f(x)$ ($y \in \mathbb{R}^n$, $x \in \mathbb{R}^n$)

want to resolve it with respect to x: if it is possible to do, then we may write

$$x = f^{-1}(y) \Rightarrow y = f(f^{-1}(y)) \quad \forall\, y \in \mathbb{R}^n$$

and

$$x = f^{-1}(f(x)) \quad \forall\, x \in \mathbb{R}^n$$

For instance, inverse to polar coordinate ~~change~~ is

$$R = R(x,y) = \sqrt{x^2+y^2}, \quad \varphi = \varphi(x,y) = \tan^{-1}(y/x)$$

Example 4: $x = u+v$, $y = u-v$, $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$ - inverse map.

Want to understand: when $f: \mathbb{R}^n \to \mathbb{R}^n$, $y = f(x)$ is invertible at least **locally** (near some point $x = x^0$ and $y^0 = f(x^0)$)

Crucial for changes of variables!!! need one-to-one correspondence between new and old variables!

Look at differentials: $dy = J(f)\big|_{x=x_0} \cdot dx$

need to treat dx as dependent variable and dy as the independent one.

$\Rightarrow$ need to solve Linear equation $dy = J(f)\,dx$ with respect to dx

$$dx = [J(f)]^{-1} dy$$

Possible to do only if J is invertible! Invertibility condition

NB! $\det J(f) \neq 0$ this det is often called Jacobian.

"Big" theorem (from Real analysis): If $\det J(f)\big|_{x=x^0} \neq 0$ then the inverse $f^{-1}: y \to x$ in a small neighbourhood of $y_0 = f(x^0)$ (local inverse function theorem).

If the existence of the inverse is known, our computations show that

$$J(f^{-1}) = J(f)^{-1}$$

Example: polar coordinates

$$\det J = \det \begin{pmatrix} \cos\varphi & -R\sin\varphi \\ \sin\varphi & R\cos\varphi \end{pmatrix} = R$$

$\Rightarrow$ locally one-to-one everywhere except of the line $R=0$

Indeed, not one-to-one near $R=0$!

All points $(0,\varphi)$ are mapped to the origin $(0,0)$!