Lecture 10: Compactness in metric spaces.

Def: A metric space (X, d) is compact iff any sequence $x_n \in X$ contains a convergent sub-sequence $x_{n_k} \to x_0 \in X$ as $k \to \infty$.

Analogously: A subset $V \subset X$ of a metric space (X, d) is compact if any sequence $x_n \in V$ has a convergent sub-sequence $x_{n_k} \to x_0$ and $x_0 \in V$.

$\Rightarrow$ [V - is compact] $\Rightarrow$ [V is closed] it is not sufficient!

Example: $X = \mathbb{R}$. Bolzano-Weierstrass theorem: any bounded sequence $x_n \in \mathbb{R}$ has a convergent subsequence.

$\Rightarrow$ [$V \subset \mathbb{R}$ is compact] $\Leftrightarrow$ [V is bounded and closed]

Example: $X = \mathbb{R}^n$ (finite-dimensional space) The same situation: compact $\Leftrightarrow$ bounded and closed. (Heine-Borel lemma).

Again, not true for infinite-dimensional case, we have seen before that a bounded and closed ball $\overline{B}_1(0)$ of $C[a,b]$ (with sup-metric) is NOT compact.

Another example: $X = l_2$ (square summable sequences)

$\overline{B}_1(0)$ is not compact

$x_n = e_n = (0, \dots \underset{n\text{th}}{1}, 0 \dots)$ - nth coordinate vector

$\|x_n - x_m\|^2 = 2 \quad \forall n, m, \ m \neq n$

$\Rightarrow \{x_n\}$ does not contain any convergent subsequence.

Typical situation for inf-dim.

Theorem (without proof):

A unit ball $\overline{B}_1(0)$ in a normed space (X, d) is compact

$\Updownarrow$

$\dim X < \infty$.

Thus, compactness is more than closed + bounded if $\dim_{\mathbb{R}} X = \infty$

How to check that X is compact?

Lemma: (X,d) is compact $\Rightarrow$ (X,d) is complete.

Proof: Let (X,d) be compact and let x_n be a Cauchy sequence in X. Then

$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon)$ such that

(*) $\quad d(x_n, x_{n+m}) < \varepsilon, \; n > N(\varepsilon), \; m \in \mathbb{N}$

X is compact $\Rightarrow \exists \; x_{n_k} \to x_0$ in X.

Let us prove that the whole sequence $x_n \to x_0$.

Indeed, (*) $\Rightarrow$

$$d(x_n, x_{n_k}) < \varepsilon, \; n > N(\varepsilon), \; n_k > n$$

Passing to the limit $k \to \infty$,

$$d(x_n, x_0) \leq \varepsilon, \; n > N(\varepsilon)$$

$\Rightarrow x_n \to x_0$. □

Only completness is not sufficient for compactness!

Def: A metric space (X,d) is totally bounded iff for any $\varepsilon > 0$ $\exists$ a finite covering of X by ε-balls $B_\varepsilon(x_1), \dots, B_\varepsilon(x_n)$, $n = n(\varepsilon)$

$$X \subset \bigcup B_\varepsilon(x_i)$$

$\{x_i\}_{i=1}^{n}$ - is a finite ε-net in X.

Lemma: (X,d) is compact $\Rightarrow$ (X,d) is totally bounded

Proof: By contradiction, assume that X is not totally bounded. Then, $\exists \; \varepsilon_0$ such that the finite ε_0-net does not exist. Let us construct a sequence $x_n \in X$ such that

$$d(x_n, x_m) \geq \varepsilon_0 \quad \forall m, n, \; m \neq n$$

Then such a sequence cannot contain any convergent subsequence $\Rightarrow$ contradiction with compactness.

Take X_1 - arbitrary, then $\{X_1\}$ is not an ε_0-net in X

$\Rightarrow \exists\, X_2 \in X \setminus B_{\varepsilon_0}(X_1) \Leftrightarrow \boxed{d(X_1, X_2) \geq \varepsilon_0}$

$\{X_1, X_2\}$ is not an ε_0-net $\Rightarrow \exists\, X_3 \in X \setminus \{B_{\varepsilon_0}(X_1) \cup B_{\varepsilon_0}(X_2)\}$

$\Rightarrow \boxed{d(X_1, X_3) \geq \varepsilon}, \boxed{d(X_2, X_3) \geq \varepsilon}$

$\vdots$

$\{X_1, \dots X_n\}$ is not an ε_0-net in $X \Rightarrow \exists\, X_{n+1} \in X \setminus \bigcup_{k=1}^{n} B_{\varepsilon_0}(X_k)$

$\Rightarrow \boxed{d(X_i, X_{n+1}) \geq \varepsilon, \quad i = 1, \dots, n}$

$\vdots$

Finally, this procedure gives the desired sequence. □

Theorem (Hausdorff criterium)

$\boxed{(X,d) \text{ is compact}} \Leftrightarrow \boxed{(X,d) \text{ is complete}}$ and $\boxed{\text{totally bounded}}$

$\Updownarrow$ closed in $\mathbb{R}^n$

$\Updownarrow$ bounded in the case of $\mathbb{R}^n$

Proof ($\Rightarrow$) is already proved.

Need to prove ($\Leftarrow$).

Let (X,d) be complete and totally bounded and let $\{X_n\} \in X$ be an arbitrary sequence.

Let us consider a 1-covering

$\boxed{X \subset B_1(X_1) \cup \dots B_n(X_n)}$ infinitely many points in finitely many balls

$\Rightarrow \exists$ ball $B_1(y_1)$ which contains infinitely many points X_n

Fix $\boxed{X_{n_1} \in B_1(y_1)}$ (the first one) and drop out X_{n_1} and all $\boxed{X_i \notin B_1(y_1)}$ from the sequence

Consider now $\varepsilon = \frac{1}{2}$ and finite covering

$\boxed{X \subset B_{1/2}(X_1) \cup \dots B_{1/2}(X_n) \quad n = n(1/2)}$

Again, at least one ball $B_{1/2}(y_2)$ contains infinitely many points

Fix $\boxed{X_{n_2} \in B_{1/2}(y_2)}$ be the first of the remaing points and drop out X_{n_2} and all $\boxed{X_i \notin B_{1/2}(y_2)}$ from the sequence.

On the kth step, we have found infinitely many points in the ball $B_{1/k}(y_k)$

and fix $\boxed{X_{n_k} \in B_{1/k}(y_k)}$ - the first one and drop out X_{n_k} and all $\boxed{X_i \notin B_{1/k}(y_k)}$

Finally, we have constructed a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a sequence of points $y_k \in X$ such that

$$x_{n_\ell} \in B_{1/k}(y_k) \quad \text{for any } \ell \geq k$$

Thus, x_{n_k} is a Cauchy sequence in X + completeness $\Rightarrow x_{n_k} \to x_0 \in X$. □

Ex.: Check that every closed subset of a compact set is compact.

Lecture 11: Continuous functions and compactness

Lemma: Let (X,d) be compact and $f: X \to \mathbb{R}$ be continuous. Then

$$\exists \min_{x \in X} f(x) \quad \text{and} \quad \max_{x \in X} f(x)$$

Proof: for the max. Let

$$A := \sup_{x \in X} f(x)$$

A priori may be infinite, but a posteriori – finite.

Then, $\exists \{x_n\} \in X$ such that

$$A = \lim_{n \to \infty} f(x_n)$$

X is compact $\Rightarrow$ $\exists$ subsequence $x_{n_k} \to x_0 \in X$.

By continuity,

$$A = \lim_{k \to \infty} f(x_{n_k}) = f\left(\lim_{k \to \infty} x_{n_k}\right) = f(x_0)$$

$\Rightarrow$ maximum exists. □

More generally:

Lemma: Let (X,d) be a compact metric space and $f: X \to Y$ be continuous. Then

$$f(X) \text{ is compact in } Y$$

Proof: Let $y_n = f(x_n)$ be an arbitrary sequence in $f(X)$.

X is compact $\Rightarrow$ $\exists\, x_{n_k} \to x_0$ in X, by continuity,

$\Rightarrow$ $y_{n_k} = f(x_{n_k}) \to y_0 := f(x_0) \in X \Rightarrow f(X)$ is compact.

□

Def: A function $f: X \to Y$ is uniformly continuous on X if $\forall \varepsilon > 0 \ \exists \ \delta = \delta(\varepsilon) > 0$ such that

$$d(f(x), f(y)) < \varepsilon \quad \text{if} \quad d(x, y) < \delta, \quad x, y \in X$$

A function $w(z)$ such that $\lim_{z \to 0} w(z) = 0$ is a modulus of continuity of f iff

$$d(f(x), f(y)) \leq w(d(x, y))$$

A function f is uniformly continuous $\Leftrightarrow \exists$ modulus of continuity (check!)

Example: $f \in C[a, b]$ is such that $f' \in C[a, b]$

Then

$$f(x) - f(y) = f'(\xi)(x - y)$$

finite increments formula

$$\Rightarrow \quad |f(x) - f(y)| \leq L |x - y| \quad \text{where} \quad L = \max_x |f'(x)|$$

$$\Rightarrow \quad w(z) = L|z| \ - \text{modulus of continuity } f.$$

Lemma: Let (X, d) be compact and let $f: X \to Y$ be continuous

Then, f is uniformly continuous on X.

Proof: By contradiction. Assume that f is not uniformly continuous. Then $\exists$ two sequences $x_n, y_n \in X$ and positive ε_0 such that

1) $d(f(x_n), f(y_n)) \geq \varepsilon$

2) $d(x_n, y_n) \to 0$ as $n \to \infty$.

Since X is compact $\Rightarrow \exists \ x_{n_k} \to x_0, \ y_{n_k} \to y_0$.

From (2) we conclude that

$$d(x_0, y_0) = 0 \Rightarrow x_0 = y_0,$$

but from (1) and continuity of f:

$$d(f(x_0), f(y_0)) \geq \varepsilon \Rightarrow f(x_0) \neq f(y_0)$$

Contradiction!

□

Example (counterexample): $X = C[a,b]$ with the sup-metric and let

$$V = \overline{B}_1(0)$$ a closed unit ball in X.

Then V is complete, but not compact.

Consider a linear function(al) on $C[-1,1]$

$$F(f) := \int_{-1}^{1} \operatorname{sgn} x \cdot f(x)\,dx$$

Then, F is continuous function on X.

Indeed, let $f_1, f_2 \in C[-1,1]$. Then

$$|F(f_1) - F(f_2)| \le \int_{-1}^{1} |f(x_1) - f(x_2)|\,dx \le \|f_1 - f_2\|_\infty \cdot \int_{-1}^{1} dx = 2\|f_1 - f_2\|_{L^\infty}.$$

(even uniformly continuous).

2) $\sup_{f \in V} F(f) = 2$. Indeed, $|F(f)| \le 2\|f\|_{L^\infty} \le 2$

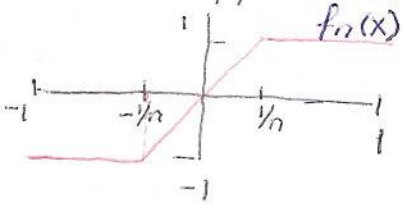
$\Rightarrow \sup_{f \in V} F(f) \le 2$. Let us prove the equality.

Hint: consider $f_0(x) = \operatorname{sgn} x$. Then, $\|f_0\|_{L^\infty} = 1$ and

$$F(f_0) = \int_{-1}^{1} \operatorname{sgn}^2 x\,dx = \int_{-1}^{1} 1\,dx = 2.$$

But f_0 is not continuous!

Solution: Approximate $f_0(x) = \operatorname{sgn} x$ by $f_n(x)$



Then f_n are continuous and

$$\|f_n\|_{L^\infty} \le 1$$

In addition,

$$|F(f_n) - F(f_0)| \le \int_{-1}^{1} |f_n(x) - f_0(x)|\,dx = \int_{-1/n}^{1/n} |f_n(x) - f_0(x)|\,dx \le 2/n.$$

Thus, $F(f_n) \to F(f_0) = 2$ as $n \to \infty$

Finally, I claim that

$\max_{f \in V} F(f)$ - does not exist!!!

By contradiction, let $f \in C[-1,1]$, $\|f\|_{L^\infty} \le 1$ be such that $\int_{-1}^{1} \operatorname{sgn} x\, f(x) = 2$. Then, since $|f(x)| \le 1$ one should have

$$\int_{0}^{1} f(x) = 1, \quad \int_{-1}^{0} f(x) = -1$$

+ f is continuous and $|f(x)| \le 1 \Rightarrow$

$f(x) \equiv 1$, $x \in [0,1]$ and $f(x) \equiv -1$, $x \in [-1,0]$

contradiction with continuity!

Lecture 12: Compactness in function spaces.

Let us consider first compactness criterium in $C[a,b]$.

Theorem (Arzela): A set $K \subset C[a,b]$ is compact iff

1) K is closed in C and is uniformly bounded

i.e. $\sup_{f\in K} \|f\|_{L^\infty} < \infty$

2) There exists a modulus of continuity w common for all $f \in K$

$|f(x)-f(y)| \le w(|x-y|)$, $x, y \in [a,b]$, $f \in K$

(= uniform continuity: $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ such that $|f(x)-f(y)| \le \varepsilon$ if $|x-y| \le \delta$ for all $f \in K$, $x, y \in [a,b]$).

Proof: $\Rightarrow$ Since K is compact, it is complete and by this reason is closed in $C[a,b]$.

Boundedness: since K is totally bounded (Hausdorff). There exists a finite 1-net $\{f_1, \dots f_n\}$. Then

$\|f\| \le \|f-f_i\| + \|f_i\| \Rightarrow \|f\|_{L^\infty} \le 1 + \max_i \|f_i\| \le C.$

Let us check the uniform continuity:

Fix $\varepsilon > 0$. By Hausdorff $\Rightarrow$ $\exists$ $\frac{\varepsilon}{4}$-net $\{f_1, \dots, f_N\}$ in K.

Every function $f: [a,b] \to \mathbb{R}$ from $C[a,b]$ is uniformly continuous $\Rightarrow$ finite number of functions $\{f_1, \dots f_N\}$ are uniformly continuous, i.e. $\exists$ $\delta = \delta(\varepsilon) > 0$ such that

$|f_i(x)-f_i(y)| \le \varepsilon/2$ if $i = 1, \dots, N$ and $|x-y| < \delta$.

Then $\forall f \in K$

$$|f(x)-f(y)| \le |f(x)-f_i(x)| + |f(x)-f_i(y)| + |f_i(x)-f_i(y)| \le$$
$$\le 2\|f-f_i\|_{L^\infty} + |f_i(x)-f_i(y)| \le 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon$$

if $|x-y| < \delta$.

$\Rightarrow$ K is uniformly continuous.

($\Leftarrow$) Assume that K is bounded closed and uniformly continuous. Need to prove that K is compact.

By Hausdorff criterium $\Rightarrow$ need to check that K is totally bounded.

Fix $\varepsilon > 0$ arbitrary and let $\delta = \delta(\varepsilon)$ be such that

$$|f(x) - f(y)| < \frac{\varepsilon}{4} \quad \text{if } |x-y| \le \delta \quad \forall f \in K$$

(uniform continuity is used)

Let us fix $\{x_i\}_{i=1}^{N} \subset [a,b]$ such that

$$[a,b] \subset \bigcup_{i=1}^{N} B_\delta(x_i)$$

- finite δ-net in $[a,b]$ (sufficient to take $x_i = a + (i-1)\delta$)

Only the total boundedness of $[a,b]$ is factualy used in the proof.

Let now $f, g \in K$, then $\forall x \in [a,b]$ $\exists\, x_i \in [a,b]$ $|x - x_i| < \delta$ $\Rightarrow$

$$|f(x) - g(x)| \le |f(x) - f(x_i)| + |g(x) - g(x_i)| + |f(x_i) - g(x_i)|$$
$$\le \frac{\varepsilon}{2} + |f(x_i) - g(x_i)|$$

$\Rightarrow$

$$\|f - g\|_{L^\infty} \le \frac{\varepsilon}{2} + \max_{i=1,..N} |f(x_i) - g(x_i)| \quad \forall f, g \in K$$

$\Rightarrow$ it is sufficient to construct the $\frac{\varepsilon}{2}$-net in K with respect to the semi-norm

$$\max_{i=1,..,N} |f(x_i)|$$

Since K is uniformly bounded $\Rightarrow$ $\exists\, M$

$$|f(x_i)| \le M, \quad i = 1, \dots, N, \quad f \in K$$

Let us split a segment $[-M, M]$ on L segments I_j, $j = 1, \dots, L$ such that

$$[-M, M] = \bigcup_j I_j \quad \text{and} \quad |I_j| < \frac{\varepsilon}{2}$$

and define

$$V_{j_1, \dots, j_N} := \{ f \in K, \quad f(x_i) \in I_{j_i}, \quad i = 1, \dots, N \}$$

some of these sets may be empty!

Then, by construction

$$K = \bigcup_{j_1, \dots, j_N \in \{1, \dots, L\}} V_{j_1, \dots, j_N}$$

and if $f, g \in V_{j_1, \dots j_N}$

$$\max_{i=1,..,N} |f(x_i) - g(x_i)| < \frac{\varepsilon}{2}.$$

Thus $\|f-g\|_{L^\infty} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ if $f, g \in V_{j_1,\dots j_N}$

$\Rightarrow$ in order to construct the ε-net in K, it is sufficient to fix ONE representative $f_{j_1,\dots j_N}$ from every non-empty class $V_{j_1\dots j_N}$.
Since we have only finitely many classes $\Rightarrow \exists$ of the finite ε-net. $\square$

Before the example – one more definition

Def: Let (X,d) be complete metric space and $V \subset X$. Then V is pre-compact if the closure $\overline{V}$ of V in X is compact.

(Ex.) Check that:

V is totally bounded $\Leftrightarrow$ $\overline{V}$ is totally bounded

Then the Hausdorff criterium reads

V is pre-compact $\Leftrightarrow$ V is totally bounded

Indeed, since a closed subset of a complete space is complete, the rest assumption of the H-criterium is automatically satisfied.

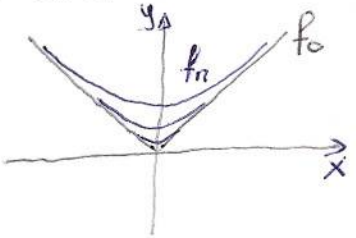
Example: Let $X = C[-1,1]$ with the sup-norm and let

$$K = \{ f, f' \in C[-1,1],\ \|f\|_{L^\infty} \le 1,\ \|f'\|_{L^\infty} \le 1 \}$$

Then, K is totally bounded = pre-compact

But K is not closed in X.

Inded, the function $f_0(x) = |x|$ which does not belong to K can be easily approximated by functions $f_n \in K$ in the C-norm:



Thus $f_0 \in \overline{K} \setminus K$ $\Rightarrow$ K is not closed.

But K is uniformly continuous:

By finite increments formula

$$f(x)-f(y)=f'(\xi)(x-y), \quad \xi\in[x,y]$$

$$\Rightarrow \quad |f(x)-f(y)| \le \max_{\xi\in[a,b]} |f'(\xi)|\cdot|x-y| \le |x-y| \quad \forall f\in K.$$

Evident fact:

$K\subset C[a,b]$ is uniformly continuous

$\Updownarrow$

$\overline{K}$ is uniformly continuous

$\Rightarrow \overline{K}$ is closed, bounded and uniformly continous

+ Arzela $\overline{K}$ is compact in $C[-1,1]$

$\Rightarrow K$ is pre-compact.

Remark: One may prove that

$$\overline{K}=\{f\in C[a,b],\ \|f\|_{\infty}\le 1,\ |f(x)-f(y)|\le|x-y|\}$$

(not necessarily $f'\in C[a,b]$, but f remains Lipschitz continuous).

K is not closed since $C[a,b]$ are "bad" (non-reflexive) spaces analogous example in $L^2[a,b]$ will be closed.

Very often in applications compactness in function spaces follows from the additional smoothness. (as in the previous example).

Compactness in the ℓ_p-spaces.

Theorem: $K\subset \ell_2$ - (space of square summable sequences) is compact iff

1) K is closed and bounded

2) $\forall \varepsilon>0 \ \exists N=N(\varepsilon)$ such that

$\forall x=(x_1,\dots x_n,\dots)\in K$

$$\sum_{i=N}^{\infty} x_i^2 \le \varepsilon$$ - uniform tail estimate.

Proof: $\Rightarrow$ K is compact $\Rightarrow$ it is closed and bounded evident.

Need to check uniform tail estimate.

By H-criterium $\forall \varepsilon > 0$ $\exists$ $\sqrt{\varepsilon}/4$-net $\{x^1, \dots, x^M\}$

Since $x^i \in \ell_2$, $i = 1, \dots, M$, there exists $N = N(\varepsilon)$ such that $\sum_{j=N}^{\infty} (x_j^i)^2 < \frac{\varepsilon}{4}$, $i = 1, \dots, M$.

Then

$$\left(\sum_{j=N}^{\infty} (x_j)^2 \right)^{1/2} \le \left(\sum_{j=N}^{\infty} (x_j^i)^2 \right)^{1/2} + \left(\sum_{j=N}^{\infty} (x - x_j^i)^2 \right)^{1/2}$$

$$\le \| x - x^i \|_{\ell_2} + \left(\sum_{j=N}^{\infty} (x_j^i)^2 \right)^{1/2} \le \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}}{2} = \sqrt{\varepsilon}.$$

($\Leftarrow$)

Let K be bounded and closed and uniform tail estimate holds. Need to prove that K is compact

By H-criterium, need to check K is totally bounded.

Fix $\varepsilon > 0$. Due to the uniform tail estimate $\exists N = N(\varepsilon)$

$$\sum_{j=N}^{\infty} (x_j)^2 \le \varepsilon^2/16 \quad \forall x \in K$$

Then $\forall x, y \in K$

$$\| x - y \|_{\ell_2} \le \left(\sum_{j=1}^{N-1} (x_j - y_j)^2 \right)^{1/2} + \left(\sum_{j=N}^{\infty} (x_j - y_j)^2 \right)^{1/2} \le$$

$$\le \left(\sum_{j=1}^{N-1} (x_j - y_j)^2 \right)^{1/2} + \left(\sum_{j=N}^{\infty} x_j^2 \right)^{1/2} + \left(\sum_{j=N}^{\infty} y_j^2 \right)^{1/2}$$

$\Rightarrow$

$$\| x - y \|_{\ell_2} \le \left(\sum_{j=1}^{N-1} (x_j - y_j)^2 \right)^{1/2} + \frac{\varepsilon}{2}$$

finite-dimensional part

Since K is uniformly bounded

$\Rightarrow |x_j| \le M$ $\forall j \in \mathbb{N}$ and $x \in K$. Split interval $[-M, M]$ by segments I_ℓ, $\ell = 1, \dots, L$ in such way that $[-M, M] = \bigcup_\ell I_\ell$ and $|I_\ell| \le \frac{\varepsilon}{2\sqrt{N-1}}$

Let $V_{\ell_1, \dots \ell_{N-1}} := \{ x \in K,\ x_j \in I_{\ell_j},\ j = 1, \dots, N-1 \}$

Then

$$K = \bigcup_{\ell_1, \dots \ell_{N-1} \in \{1, \dots, L\}} V_{\ell_1, \dots, \ell_{N-1}}.$$

And, for any $x, y \in V_{e_1,\dots,e_{N-1}}$

$$\left(\sum_{j=1}^{N-1} |x_j - y_j|^2\right)^{1/2} \le \frac{\varepsilon}{2} \Rightarrow \|x-y\|_{\ell_2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

To construct the desired ε-net, we only need to fix one representative from every non-empty class $V_{e_1,\dots,e_{N-1}}$. □

Example: typical example of a compact set in ℓ_2:

$$K := \left\{ x \in \ell_2, \ \sum_{j=1}^{\infty} j^2 x_j^2 \le 1 \right\}$$

Indeed, the uniform tail estimate holds since

$$\sum_{j=N}^{\infty} x_j^2 \le \frac{1}{N^2} \sum_{j=N}^{\infty} j^2 x_j^2 \le \frac{1}{N^2} \sum_{j=1}^{\infty} j^2 x_j^2 \le 1/N^2$$

+ K is bounded.

Check that K is closed: let

$x^n \to x^0$ in ℓ_2 and $x^n \in K$

Need to prove: $x^0 \in K$.

$$\sum_{j=1}^{M} j^2 (x_j^n)^2 \le \sum_{j=1}^{\infty} j^2 (x_j^n)^2 \le 1 \qquad \forall\, x^n \in K$$

finitely many terms + $x_j^n \to x_j^0$ for any j

Passing to the limit $n \to \infty$

$$\sum_{j=1}^{M} j^2 (x_j^0)^2 \le 1$$

Passing to the limit $M \to \infty$

$$\sum_{j=1}^{\infty} j^2 (x_j^0)^2 \le 1$$

$\Rightarrow x^0 \in \ell_2$ and K is closed

By the previous theorem K is compact.

Remark: Factually, K is a unit ball in a Sobolev space $W_2^1[a,b] = \{ f, f' \in L^2[a,b] \}$

Again compactness follows from additional smoothness.