Lecture 7: Completness and completion.

Examples: $(\mathbb{Q}, |\cdot|)$, $(\mathbb{R}, |\cdot|)$, $([a,b], |\cdot|)$, $((a,b), |\cdot|)$ are metric spaces. In particular,

$X = (a,b)$ with stanard metric. What is the difference between X and $\widetilde{X}$?

If we consider $X \subset \mathbb{R}$ and $\widetilde{X} \subset \mathbb{R}$ then X is open and $\widetilde{X}$ is closed - the difference is clear.

But let us consider $(X, |\cdot|)$ and $(\widetilde{X}, |\cdot|)$ as METRIC spaces. Then X, and $\widetilde{X}$ are BOTH open and closed

$\Rightarrow$ no difference?

But every continuous function $f: \widetilde{X} \to \mathbb{R}$ is bounded and achieves it's minimum and maximum. That is not true for the space X!

$\Rightarrow$ There is an intrinsic difference between X and $\widetilde{X}$!

How to describe this difference?

$\mathbb{Q}$ - is "bad" from the calculus point of view and $\mathbb{R}$ is "good" - where is the difference?

$\Rightarrow$ completness property (Axiom C for $\mathbb{R}$)

Def $x_n \in \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$ iff $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ such that

$$|x_n - x_{n+m}| < \varepsilon \quad n \geq N(\varepsilon),\ m \in \mathbb{N}$$

Def (X,d) is a metric space $x_n \in X$ is a Cauchy sequence iff $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ such that

$$d(x_n, x_{n+m}) < \varepsilon \quad n \geq N(\varepsilon),\ m \in \mathbb{N}$$

Reformulation: x_n is a Cauchy sequence in X iff $\forall \varepsilon > 0$ $\exists x_\varepsilon \in X$ and $N = N(\varepsilon)$ such that

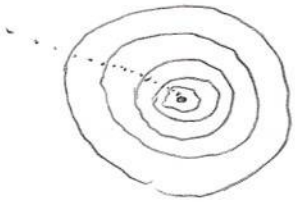
$$x_n \in B_\varepsilon(x_\varepsilon) \text{ if } n > N(\varepsilon)$$

Please, check the equivalence as an Ex.!

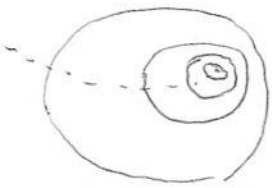
Reminder: $x_n \to x_0$ in X iff $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$

$$x_n \in B_\varepsilon(x_0) \text{ if } n > N(\varepsilon)$$

$\Rightarrow$ for convergence we need $\boxed{x_\varepsilon \equiv x_0}$ for all $\varepsilon > 0$ and for the Cauchy sequence x_ε may be different for different ε!



Convergence
(con-centric balls)



Cauchy sequence
(non-concentric balls)

Clearly: Every convergent sequence is a Cauchy sequence. But a Cauchy sequence is not necessarily convergent.

Example: $X = (0,1)$, $x_n = \frac{1}{n}$ is a Cauchy sequence in X, but it converges to $x_0 = 0 \notin X$. So, x_n does not have any limit in X.

Def A metric space (X,d) is complete iff any Cauchy sequence $x_n \in X$ has a limit $x_0 \in X$.
A complete normed space = Banach space

Reminder: $X = \mathbb{R}$ here we have the completeness axiom (Axiom C, see RA1)

for any Bounded set $B \subset \mathbb{R}$ there exists $\sup B$ (uses order structure on $\mathbb{R}$)

then, we have $\Leftrightarrow$

Cauchy Theorem: Any Cauchy sequence in $\mathbb{R}$ has a limit. (completeness)

$\Leftrightarrow$

Weierstrass theorem: Any monotone bounded sequence has a limit

$\Leftrightarrow$

Bolzano-Weierstrass theorem: Any bounded sequence in $\mathbb{R}$ contains a convergent subsequence (compactness)

All these statements are equivalent in $\mathbb{R}$. The reason $\mathbb{R}$ is "too simple" and everything is "mixed" here.
No equivalence for more general metric spaces!!!

Example: Let $X = C[a,b]$ be a space of continuous functions $f: [a,b] \to \mathbb{R}$ with the sup-norm

$$\|f\|_{L^\infty} := \max_{x\in[a,b]} |f(x)|$$

Theorem: $(C[a,b], \|\cdot\|_{L^\infty})$ is complete. I.e., $C[a,b]$ with the sup-norm is a BANACH space.

Proof: Let $f_n \in X$ be a Cauchy sequence in X. We need to construct a limit function f for that sequence. Let $x_0 \in [a,b]$ be arbitrary. Consider a sequence of numbers $f_n(x_0)$. Since

$$|f_n(x_0) - f_{n+m}(x_0)| \le \max_x |f_n(x) - f_{n+m}(x)| = \|f_n - f_{n+m}\|_{L^\infty}$$

and f_n is a Cauchy sequence in $C[a,b] \Rightarrow \{f_n(x_0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. But $\mathbb{R}$ is complete

$\Rightarrow \forall x_0 \in [a,b] \ \exists \ f(x_0) := \lim_{n\to\infty} f_n(x_0)$

Need to check: 1) the limit is uniform 2) f is a continuous function.

(1) Since f_n is a Cauchy sequence $\Rightarrow \forall \varepsilon > 0 \ \exists N = N(\varepsilon)$

$\forall x \in [a,b]$ $|f_n(x) - f_{n+m}(x)| \le \|f_n - f_{n+m}\|_{L^\infty} < \varepsilon$, $n \in \mathbb{N}$, $n \ge N(\varepsilon)$

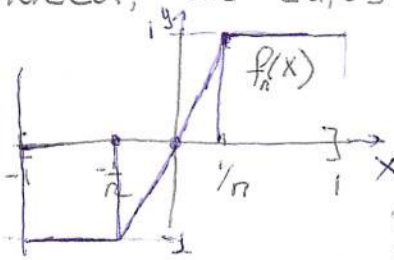
Passing to the limit $m \to \infty$, we have

(uniform convergence is proved) $|f_n(x) - f(x)| < \varepsilon$, $n \ge N(\varepsilon)$ $\forall x \in [a,b]$

(2) A uniform limit of continuous functions is a continuous function. Please, check! □

More Example (BAD NEWS): $X = C[a,b]$ with the integral norm $\|f\|_{L^1} := \int_a^b |f(x)|\,dx$ – is NOT complete!

Indeed, let $[a,b] = [-1,1]$ and let



Then $f_n \xrightarrow{\text{(point-wise)}} f(x) = \operatorname{sgn} x$ – not continuous, but integrable

$$\|f_n - f\|_{L^1} := \int_{-1}^{1} |f_n(x) - f(x)|\,dx =$$

$$= \int_{-1/n}^{1/n} |f_n(x) - f(x)|\,dx \le 2 \int_{-1/n}^{1/n} dx \le 4/n \to 0$$

$\Rightarrow f_n$ is a Cauchy sequence in the L^1-norm, but its limit $f \notin C[a,b] \Rightarrow$ no limits in $C[a,b]$

$\Rightarrow (C[a,b], \|\cdot\|_{L^1})$ – is not complete.

Lecture 8: Completeness and completion (continued)

Example: Let $X = \ell_2$ (square sumable sequences)

$$\|x\|^2_{\ell^2} = \sum_{i=1}^{\infty} x_i^2$$

Lemma: $(\ell_2, \|\cdot\|_{\ell_2})$ is a Banach space.

Proof: Let $x^n = (x_1^n, x_2^n, \dots)$ be a Cauchy sequence in ℓ_2

Need to prove $\exists\, x^0 \in \ell_2$ such that $x^n \to x^0$ in ℓ_2.

Using the inequality

$$|x_i^n - x_i^{n+m}| \le \|x^n - x^{n+m}\|_{\ell^2} \quad i \in \mathbb{N}$$

and that x^n is a Cauchy sequence in ℓ_2, we see that $\{x_i^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ $\Rightarrow$

$\forall i \in \mathbb{N}$ $\quad x_i^0 := \lim_{n \to \infty} x_i^n$ - exists (coordinate-wise limit)

Need to prove

① $x^0 = (x_1^0, \dots, x_i^0) \in \ell_2$

② $\|x^n - x^0\| \to 0$, $n \to \infty$

① Since x^n is a Cauchy sequence in ℓ_2 and

$$|\,\|x^n\| - \|x^{n+m}\|\,| \le \|x^n - x^{n+m}\|$$

$\Rightarrow$ $\|x^n\|$ is a Cauchy sequence in $\mathbb{R}$ $\Rightarrow$ $\|x^n\|$ is convergent

$\Rightarrow$ $\|x^n\|$ is a bounded sequence

$\Rightarrow$ $\sum_{i=1}^{N} (x_i^n)^2 \le \|x^n\|^2_{\ell^2} \le C \quad \forall N$

Passing to the limit $n \to \infty$

$$\sum_{i=1}^{N} (x_i^0)^2 \le C$$

Passing to the limit $N \to \infty$

$$\sum_{i=1}^{\infty} (x_i^0)^2 \le C \Rightarrow$$

$$x^0 \in \ell_2$$

(2) $\forall \varepsilon > 0 \quad \exists N = N(\varepsilon)$

$$\sum_{i=1}^{M} (x_i^n - x_i^{n+m})^2 \le \|X^n - X^{n+m}\|_{\ell^2}^2 \le \varepsilon, \quad n \ge N(\varepsilon), \ M \in \mathbb{N}, \ m \in \mathbb{N}$$

Passing to the limit $m \to \infty \Rightarrow$

$$\sum_{i=1}^{M} (x_i^n - x_i^0)^2 \le \varepsilon \qquad M \in \mathbb{N}, \ n \ge N(\varepsilon)$$

Passing to the limit $M \to \infty$

$$\sum_{i=1}^{\infty} (x_i^n - x_i^0)^2 := \|X^n - X^0\|_{\ell_2}^2 \le \varepsilon$$

$\Rightarrow X^n \to X^0$ in ℓ^2. $\square$

Warning: if $x_i^n \to x_i^0$ as $n \to \infty$, $i = 1, 2, \dots, M$ then, obviously,

$$\sum_{i=1}^{M} x_i^n \to \sum_{i=1}^{M} x_i^0$$

But that can be WRONG for $M = \infty$!

Example: $x_i^n = 0$, $n \ne i$, $x_i^i = 1$. Then, evidently

$$x_i^0 = \lim_{n \to \infty} x_i^n = 0, \text{ but}$$

$$\sum_{i=1}^{\infty} x_i^n = 1 \quad \forall n \in \mathbb{N}$$

Never forget this example !!!

Something more is needed to justify the change of limits

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} x_i^n = \sum_{i=1}^{\infty} \left(\lim_{n \to \infty} x_i^n \right)$$

– it is not automatically true!

(Ex.) Check that ℓ_p, $1 \le p \le \infty$ are complete.

Model example of a non-compete metric space

Let (X, d) be a (complete) metric space and let $V \subset X$ be a subset of X which is NOT closed. Then the metric space (V, d) is not complete. Indeed, let $x_n \in V$ be such that $x_n \to x_0 \in X$ and $x_0 \notin V$ (V is not closed). Then, x_n is a Cauchy sequence in V which does not have any limit in V. $\Rightarrow (V, d)$ is not complete.

In a fact, ANY non-complete space can be obtained in a such way.

Def.: A space $(\tilde{X}, \tilde{d})$ is a completion of a metric space (X, d) if

1) $X \subset \tilde{X}$ and $\tilde{X}$ is complete
2) X is dense in $\tilde{X}$
3) $\tilde{d}(x, y) = d(x, y)$ if $x, y \in X$.

Completion theorem: for any metric space (X, d) there exists a completion $(\tilde{X}, \tilde{d})$. This completion is unique up to the isometric one-to-one map.

I.e. if $(\tilde{X}, \tilde{d})$ and $(\tilde{\tilde{X}}, \tilde{\tilde{d}})$ be two completions of (X, d), then there exists a one-to-one map $\tau: \tilde{X} \to \tilde{\tilde{X}}$ such that

a) $\tau|_X = Id$ $(\tau(x) = x \quad \forall x \in X)$

b) $\tilde{\tilde{d}}(\tau(x_1), \tau(x_2)) = \tilde{d}(x_1, x_2)$, $x_1, x_2 \in \tilde{X}$.

Without the formal proof in this course!

Idea: to construct $\tilde{X}$ by considering any Cauchy sequences in X as points in $\tilde{X}$.

$\Rightarrow$ There is a natural way to produce a complete metric space (minimal complete extension) of a given metric space $\Rightarrow$ it is sufficient to study only complete metric spaces.

Warning: A concrete (simplest) realization of a completion is, nevertheless, important!

Examples: 1) the completion of $\mathbb{Q}$ is $\mathbb{R}$

2) the completion of $\mathbb{Q}^n$ is $\mathbb{R}^n$.

3) Let $X = P[a,b]$ - space of all polynoms on $[a,b]$

$$f(x) = a_n x^n + \dots + a_0, \quad n \in \mathbb{N}$$

Then $(P[a,b], \|\cdot\|_{L^\infty})$ is not complete

$e^x = \lim\limits_{n\to\infty} \sum\limits_{i=0}^{n} \frac{x^i}{i!}$, but e^x is not a polynom!

What is the completion of $P[a,b]$?

By the Weierstrass theorem, $P[a,b]$ is dense in $C[a,b]$ (any continuous function can be obtained as a uniform limit of polynoms)

$\Rightarrow$ the completion of $P[a,b]$ is $C[a,b]$ (of course, with respect to the sup-metric).

Corollary: The space $(C[a,b], \|\cdot\|_{L^\infty})$ is separable.

Proof: Indeed, the polynoms with rational coefficients are dense in $P[a,b]$ and there are countably many of such polynoms. Since $P[a,b]$ are dense in $C[a,b]$ $\Rightarrow$ polynoms with rational coefficients are dense in $C[a,b]$ □

General fact: A closure of a separable set is separable. Please, check!

Lecture 9: Lebesgue and Sobolev spaces.

As we see, the space $(C[a,b], \|\cdot\|_{L^1})$ with $\|f\|_{L^1} = \int_a^b |f(x)|\,dx$ is not complete.

Analogously, the space $(C[a,b], \|\cdot\|_{L^p})$ where $\|f\|_{L^p} = \left(\int_a^b |f(x)|^p\right)^{1/p}$, $1 \le p \le \infty$ is not complete.

Def.: For any $1 \le p < \infty$, $L_p([a,b])$ is a completion of $C[a,b]$ with respect to the metric $\|\cdot\|_{L_p}$. Lebesgue space

Corollary: $L_p([a,b])$ is separable, $1 \le p < \infty$.

Indeed, $C[a,b]$ is separable in sup-metric $\Rightarrow$ it is separable in the L_p-metric, $C[a,b]$ is dense in $L_p[a,b]$ by definition $\Rightarrow$ $L_p[a,b]$ is separable.

On the abstract level, - def of $L_p([a,b])$ is OK.

Concrete realization - more difficult question

Are the elements of $L_p([a,b])$ (discontinuous) functions?

Answer - Yes

Difficulties: 1) $\|\cdot\|_{L_p}$ is no more a norm on the space of discontinuous functions

e.g. a function which is not zero only on finitely many points have zero L_p-norm

$\Rightarrow$ elements of $L_p([a,b])$ are defined only up a "zero measure" set.

2) The Riemann integral is not sufficient for the proper definition of such function

$\Rightarrow$ we need a measure theory and Lebesgue integration.

If that is done one can describe $L_p([a,b])$ as a set of functions $f: [a,b] \to \mathbb{R}$ (up to the "zero measure" set) such that

$$\int_a^b |f(x)|^p dx < \infty$$

In a fact, all Riemann integrable discontinuous functions belong to that class + some more functions.

Sobolev spaces (very briefly).

Def. Let $C^n([a,b])$ be the space of all n-time continuously differentiable functions ($f^{(n)}(x)$ is continuous on $[a,b]$)

Natural norms:

1) the sup-norm

$$\|f\|_{C^n([a,b])} := \|f\|_{C[a,b]} + \|f'\|_{C[a,b]} + \dots + \|f^{(n)}\|_{C[a,b]}$$

complete metric space + linear structure = Banach space — check!

2) Integral norms:

$$\|f\|^p_{W_p^n([a,b])} := \|f\|^p_{L^p} + \|f'\|^p_{L^p} + \dots + \|f^{(n)}\|^p_{L^p}$$

not complete

Def: A Sobolev space $W_p^n([a,b])$ is a completion of the space $C^n([a,b])$ with respect to $\|\cdot\|_{W_p^n}$.

Again: on the abstract level - def is OK

Concrete realization: much more difficult in addition, we need to define a "derivative" of a discontinuous function $\Rightarrow$ theory of distributions and Sobolev spaces

Sobolev spaces made a Revolution in PDEs in XX century spaces of discontinuous functions, singularities and "generalized" derivatives are now unavoidable here

Important: By definition $C^n([a,b])$ is dense in $W_p^n([a,b])$ and since C^n is separable $W_p^n([a,b])$ are also separable.