Coursework 1, Function Spaces, 2007/2008, MS310/320.

BSc: Problem 1-3 and 5. MMath: Problem 1-4.

Please hand-in by Friday October 20.

At all problems, please write down the relevant definitions in your answer.

Problem 1: Recall that for any subset $X \subset \mathbb{R}$,

$$L^p(X) = \left\{ f : X \to \mathbb{R} : \int_X |f(t)|^p dt < \infty. \right\}$$

Let $f(x) = (\frac{1}{x})^\alpha$ for some fixed $\alpha \in [0, \infty)$.

a) For which p (in terms of α) does f belong to $L^p((0,1))$? [2]

b) Show that $L^q((0,1)) \subset L^p((0,1))$ if $1 \leq p < q \leq \infty$. [3]

c) Repeat a) for $L^p((1,\infty))$. [3]

d) What is the bigger space when $1 \leq p < q \leq \infty$: $L^p((1,\infty))$ or $L^q((1,\infty))$? (**Think twice! and motivate your answer**) [3]

Problem 2: Two norms $\| \; \|_a$ and $\| \; \|_b$ are **equivalent** if there is $K \geq 1$ such that for all $x, y \in X$,

$$\frac{1}{K}\|x\|_a \leq \|x\|_b \leq K\|x\|_a.$$

a) Show that on $X = \mathbb{R}^2$, the norms $\| \; \|_\infty$, $\| \; \|_1$ and $\| \; \|_2$ are all equivalent. [3]

b) Show that if $\| \; \|_a$ and $\| \; \|_b$ are equivalent norms, then any convergent sequence in $(X, \| \; \|_a)$ must also be convergent in $(X, \| \; \|_b)$, and the limits are the same. [4]

c) Let $X = C([0,1])$. Are the norms $\| \; \|_1$ and $\| \; \|_\infty$ equivalent? Justify your answer. [4]

Problem 3: a) Show that a Lipschitz function between two metric spaces is uniformly continuous. [3]

b) Suppose that $f : [0,1] \to \mathbb{R}$ is continuously differentiable. Show that f is Lipschitz with Lipschitz constant

$$L = \sup_{x \in [0,1]} |f'(x)|.$$

Also show that no smaller Lipschitz constant works. [4]

c) Let

$$X = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$$

and

$$Y = \{f : [0,1] \cap \mathbb{Q} \to \mathbb{R} : f \text{ is continuous}\},$$

both with $\| \|_\infty$-norm.

Show that if $f \in Y$ and is **uniformly** continuous, then there is a unique **extension** $f_* \in X$ such that $f_*(x) = f(x)$ for all $x \in [0,1] \cap \mathbb{Q}$.

Hint: For each $x \in [0,1]$, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset [0,1] \cap \mathbb{Q}$ such that $x_n \to x$. Define $f_*(x) = \lim_{n\to\infty} f(x_n)$. Then show that this limit indeed exists, and is the same for every sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}} \subset [0,1] \cap \mathbb{Q}$ that converges to x. [5]

d) Give a function $g \in Y$ that does not have an extension to X. [2]

Problem 4: This problem aims at showing that

$$A := \{f \in X : f \text{ has Lipschitz constant } \leq 1\}$$

is a compact subset of X in $\| \|_\infty$-norm.

Recall that $\mathbb{Q} \cap [0,1]$ is a dense countable subset of $[0,1]$. Let $\{q_n\}_{n\in\mathbb{N}}$ be a denumeration of $\mathbb{Q} \cap [0,1]$.

a) Let $\{f_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in A. Fix $x = q_1$. Show that there is a subsequence $\{n_k^1\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\{f_{n_k^1}(q_1)\}_{k\in\mathbb{N}}$ converges. [3]
b) Find a subsequence $\{n_k^2\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that both $\{f_{n_k^2}(q_1)\}_{k\in\mathbb{N}}$ and $\{f_{n_k^2}(q_1)\}_{k\in\mathbb{N}}$ converge. [2]

Continuing this way, one can construct a sequence, say $\{m_k\}_{k\in\mathbb{N}}$, such that $\{f_{m_k}\}_{k\in\mathbb{N}}$ converges pointwise at every $q \in \mathbb{Q} \cap [0,1]$. Bonus marks if you explain precisely how such a sequence $\{m_k\}_{k\in\mathbb{N}}$ is to be found.

c) Let f be the pointwise limit of $\{f_{m_k}\}_{k\in\mathbb{N}}$. Show that f is Lipschitz on $[0,1] \cap \mathbb{Q}$, with a Lipschitz extension $f_* : [0,1] \to \mathbb{R}$ (both with Lipschitz constant ≤ 1). [4]
d) Show that f has an extension $f_* \in X$, and show that $f_* \in A$. Hence show that A is indeed compact subset of X. [5]

Problem 5: a) Show that $(\ell^\infty, \| \|_\infty)$ is a complete space. [5]

b) Let

$$\|x\|_* = \sum_{n=1}^{\infty} 2^{-n} |x_{n+1} - x_n|$$

Show that $\| \|_*$ is a semi-norm on ℓ^∞ (check the axioms explicitly). Explain why the factors 2^{-n} are needed. [4]

c) Find a combination of $\| \|_\infty$ and $\| \|_*$ to create a proper norm $\| \|_\flat$ such that $\| \|_* \leq \| \|_\flat$. [3]

d) Find a function $d : \ell^\infty \times \ell^\infty \to \mathbb{R}$ that satisfies all axioms of a metric, except that it is non-symmetric: there are $d(x,y) \neq d(y,x)$ for some $x, y \in \ell^\infty$. [2]

Solutions.

Problem 1:
a) Here $f(x) = x^{-\alpha}$, so $\int_0^1 |f(t)|^p dt = \int_0^1 x^{-\alpha p} dt = \frac{1}{1-\alpha p} < \infty$, provided $\alpha p < 1$. Otherwise the integral diverges. Therefore $f \in L^p((0,1))$ if and only if $\alpha < 1/p$.
b) Take $f \in L^q((0,1))$ and $X_- = \{x \in (0,1) : f(x) \leq 1\}$ and $X_+ = \{x \in (0,1) : f(x) > 1\}$. Then

$$\int_0^1 |f(t)|^p dt = \int_{X_-} |f(t)|^p dt + \int_{X_+} |f(t)|^p dt \leq \int_{X_-} 1 dt + \int_{X_+} |f(t)|^q dt \leq 1 + \int_0^1 |f(t)|^q dt < \infty.$$

Therefore $f \in L^p((0,1))$. Since this holds for every $f \in L^q((0,1))$, $L^q((0,1)) \subset L^p((0,1))$.
c) $\int_1^\infty |f(t)|^p dt = \int_1^\infty x^{-\alpha p} dt = \frac{1}{\alpha p - 1} < \infty$, provided $\alpha p > 1$. Otherwise the integral diverges. Therefore $f \in L^p((0,1))$ if and only if $\alpha > 1/p$.
d) Neither of $L^p((1,\infty))$ or $L^q((1,\infty))$ is contained in the other. First, take $\frac{1}{q} < \alpha < \frac{1}{p}$ (or $0 < \alpha < \frac{1}{p}$ if $q = \infty$). Then $f(x) = x^{-\alpha}$ belongs to $L^q((1,\infty))$ but not to $L^p((1,\infty))$, see part c). next take $g(x) = (x-1)^{-\alpha}$ if $x \in (1,2)$ and $g(x) = 0$ otherwise. Then g belongs to $L^p((1,\infty))$ but not to $L^q((1,\infty))$, see part a).

Problem 2:
a)

$$\begin{aligned} \|x\|_\infty &= \max\{|x_1|, |x_2|\} \leq |x_1| + |x_2| = \|x\|_1 \\ &\leq \sqrt{|x_1|^2 + |x_2|^2} + \sqrt{|x_1|^2 + |x_2|^2} = 2\|x\|_2 \\ &\leq 2\sqrt{|x_1|^2 + |x_2|^2} \leq 2\sqrt{2\max\{|x_1|^2, |x_2|^2\}} = 2\sqrt{2}\|x\|_\infty \end{aligned}$$

so all three norms are equivalent, for example with the constant $K = 2\sqrt{2}$.
b) Suppose $K \geq 1$ is such that $\frac{1}{K}\|x\|_a \leq \|x\|_b \leq K\|x\|_a$ for all $x \in X$. Take $\varepsilon > 0$ arbitrary and assume that $x_n \to x$ is $\| \ \|_a$. Hence there is N such that for all $n \geq N$, $\|x_n - x\|_a < \varepsilon/K$. But then also $\|x_n - x\|_b \leq K\|x_n - x\|_a < K\varepsilon/K = \varepsilon$, so $x_n \to x$ in $\| \ \|_b$.
c) Suppose $K \geq 1$ is such that $\frac{1}{K}\|f\|_1 \leq \|f\|_\infty \leq K\|f\|_1$ for all $f \in C([0,1])$. Take $f_n(x) = x^n$. Then $1 = \|f_n\|_\infty \leq K\|f_n\|_1 = \frac{K}{n+1}$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we get a contradiction.

Problem 3:
a) $f : (X,d) \to (X',d')$ is Lipschitz if there is a constant L (called **Lipschitz constant**) such that $d'(f(x), f(y)) \leq L d(x,y)$ for all $x, y \in X$.

Take $\varepsilon > 0$ arbitrary and $\delta = \varepsilon/L$. Then for every $x, y \in X$ such that $d(x,y) < \delta$ we have $d'(f(x), f(y)) \leq L d(x,y) < L\delta = L\varepsilon/L = \varepsilon$.

Hence f is uniformly continuous.
b) If $x, y \in [0,1]$, then by the Mean Value Theorem, there is $\xi \in [x,y]$ such that $|f(x) - f(y)| = |f'(\xi)(x-y)| \leq \sup_{t \in [0,1]} |f'(t)||x-y|$, so $L = \sup_{t \in [0,1]} |f'(t)|$ is a Lipschitz constant.

If there was a smaller Lipschitz constant, say L_0. then there is $\xi \in [0,1]$ and $L_1 \in \mathbb{R}$ such that $L_0 < L_1 < |f'(\xi)| \leq \sup_{t \in [0,1]} |f'(t)|$. Since f' is continuous, we can take $\varepsilon > 0$ so small that $|f'(t)| \geq L_1$ for all $t \in (\xi - \varepsilon, t + \varepsilon)$. Then for any $\xi - \varepsilon < x < y < \xi + \varepsilon$, we have

$$|f(y) - f(x)| = |\int_x^y f'(t) dt| \geq \int_x^y |f'(t)| dt \geq L_1|y - x| > L_0|y - x|$$

so L_0 is not a Lipschitz constant.
c) Take $\varepsilon > 0$ arbitrary and $\delta > 0$ such that for all $s, t \in [0,1] \cap \mathbb{Q}$ with $|s-t| < \delta$, we have $|f(s) - f(t)| < \varepsilon$.

Since $x_n \to x$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \delta/2$. So in particular, for all $n, m \geq N$, $|x_n - x_m| \leq |x_n - x| + |x_m - x| < \delta/2 + \delta/2 = \delta$.

Therefore $|f(x_n) - f(x_m)| < \varepsilon$. In other words, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, $\{f(x_n)\}_{n\in\mathbb{N}}$ must converge; call the limit $f_*(x)$.

Now assume by contradiction that there is another sequence $\tilde{x}_n \to x$ such that $f(\tilde{x}_n) \to a \neq f_*(x)$. Take $\varepsilon = |f_*(x) - a|/3$ and let take $\delta > 0$ so small that for all $s, t \in [0,1] \cap \mathbb{Q}$ with $|s-t| < \delta$, we have $|f(s) - f(t)| < \varepsilon$. Next take $N \in \mathbb{N}$ be so large that for all $n \geq N$, both $|x_n - x| < \delta$, $|f(x_n) - f_*(x)| < \varepsilon$ and $|\tilde{x}_n - x| < \delta/2$, $|f(\tilde{x}_n) - a| < \varepsilon$. Then $|x_n - \tilde{x}_n| \leq |x_n - x| + |\tilde{x}_n - x| < \delta/2 + \delta/2 = \delta$. Therefore

$$|f_*(x) - a| \leq |f_*(x) - f(x_n)| + |f(x_n) - f(\tilde{x}_n)| + |f(\tilde{x}_n) - a| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon = |f_*(x) - a|,$$

a contradiction. So for every sequence $\tilde{x}_n \to x$, $\{f(\tilde{x}_n)\}_{n\in\mathbb{N}}$ must have the same limit.
d) Take $g(x) = 1$ for $x \leq 1/\pi$ and $g(x) = 0$ for $x > 1/\pi$. Then g is continuous as function from $[0,1] \cap \mathbb{Q}$ to $\mathbb{R}$, but discontinuous as function from $[0,1]$ to $\mathbb{R}$. Hence g has no continuous extension g_*. (Note that g is not uniformly continuous on $[0,1] \cap \mathbb{Q}$!)

Problem 4:
a) Take q_1 and compute $|f_n(q_1)| = |f(q_1) - q_n(0)| \leq 1 \cdot |q_1 - 0| \leq 1$, so $\{f_n(q_1)\}_{n\in\mathbb{N}} \subset [-1,1]$. This interval is compact, so there is a subsequence $\{n_k^1\}_k \subset \mathbb{N}$ such that $\{f_{n_k^1}(q_1)\}_{k\in\mathbb{N}}$ converges. Call the limit $f(q_1)$.
b) Likewise $|f_n(q_2)| = |f(q_2) - q_n(0)| \leq 1 \cdot |q_1 - 0| \leq 1$, so $\{f_{n_k^1}(q_1)\}_{k\in\mathbb{N}} \subset [-1,1]$. This interval is compact, so there is a subsequence $\{n_k^2\}_k \subset \{n_k^1\}_{k\in\mathbb{N}}$ such that $\{f_{n_k^2}(q_2)\}_{k\in\mathbb{N}}$ converges. (Call the limit $f(q_1)$.) Because $\{n_k^2\}_k$ is a subsequence of $\{n_k^1\}_{k\in\mathbb{N}}$, also $\{f_{n_k^2}(q_1)\}_{k\in\mathbb{N}}$ converges.
c) Take $q_a, q_b \in [0,1] \cap \mathbb{Q}$ arbitrary. Since the function $x \mapsto |x|$ is continuous, we get

$$|f(q_a) - f(q_a)| = |\lim_k f_{m_k}(q_a) - \lim_k f_{m_k}(q_b)| = \lim_k |f_{m_k}(q_a) - f_{m_k}(q_b)| \leq \lim_k 1 \cdot |q_a - q_b|. = 1 \cdot |q_a - q_b|,$$

so f is Lipschitz on $[0,1] \cap \mathbb{Q}$.

Lipschitz functions are uniformly continuous, see Problem 2a. Therefore, f has a continuous extension $f_* : [0,1] \to \mathbb{R}$.

To show that f_* is also Lipschitz, take $x, y \in [0,1]$ and sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $[0,1] \cap \mathbb{Q}$ that converge to x and y respectively. Then

$$|f_*(x) - f_*(y)| = |\lim_n f(x_n) - \lim_n f(y_n)| = \lim_n |f(x_n) - f(y_n)| \leq \lim_n 1 \cdot |x_n - y_n| = |x - y|.$$

Therefore f_* is indeed Lipschitz with Lipschitz constant 1.
d) Let $\{f_n\}_{n\in\mathbb{N}}$ be any subsequence in A. It is given (cf. parts a. and b. and the remark after it) that there is a subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$ so that $\{f_{m_k}(q)\}_{k\in\mathbb{N}}$ converges for every $q \in [0,1] \cap \mathbb{Q}$. By part c), the pointwise limit is Lipschitz, and has a Lipschitz extension f_* to $[0,1]$. Moreover $f_*(0) = \lim_k f_{m_k}(0) = \lim_k 0 = 0$. Hence $f_* \in A$.

It remains to show that $f_{m_k} \to f_*$ in $\| \ \|_\infty$. Take $\varepsilon > 0$ arbitrary. Let $\delta < \varepsilon/4$. There is a finite collection Q of $q \in [0,1] \cap \mathbb{Q}$ such that every $x \in [0,1]$, there is $q = q(x) \in Q$ such that

$|x-q| < \delta$. As Q is finite, there is N such that for all $k \geq N$, and all $q \in Q$, $|f_{m_k}(q) - f(q)| < \varepsilon/4$. Now, because all the f_{m_k} are Lipschitz with the same constant 1, we have for $k \geq N$:

$$\begin{aligned} |f_*(x) - f_{m_k}(x)| &\leq |f_*(x) - f_*(q(x))| + |f_*(q(x)) - f_{m_k}(q(x))| + |f_{m_k}(q(x)) - f_{m_k}(x)| \\ &\leq |x - q(x)| + |f_*(q(x)) - f_{m_k}(q(x))| + |q(x) - x| \\ &\leq \delta + \varepsilon/4 + \delta \leq 3\varepsilon/4 < \varepsilon. \end{aligned}$$

This shows the uniform convergence.

Problem 5: a) Let $\{x^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in ℓ^∞. The proof consists of three steps:
i) find a candidate limit a.
ii) show that $a \in \ell^\infty$, and
iii) show that indeed $x^n \to a$ in $\| \ \|_\infty$.
To prove i), observe that since $\{x^n\}_{n\in\mathbb{N}}$ is Cauchy, also each of the coordinate sequences $\{x_k^n\}_{n\in\mathbb{N}}$ (for fixed k) is a Cauchy sequence in $\mathbb{R}$. But $\mathbb{R}$ is complete, so x_k^n converges to some a_k as $n \to \infty$. Let $a = (a_k)_{k=1}^\infty$ be the candidate limit.
ii) Given $\varepsilon > 0$, there exists N such that for all $m, n \geq N$, and all $K \geq 1$,

$$\sup_{k=1,\dots,K} |x_k^n - x_k^m| \leq \sup_{k\geq 1} |x_k^n - x_k^m| \leq \varepsilon/2.$$

First let $m \to \infty$ to obtain $\sup_{k=1,\dots,K} |x_k^n - a_k| \leq \varepsilon/2$, and then let $K \to \infty$ to obtain

$$\sup_{k\geq 1} |x_k^n - a_k| \leq \varepsilon/2 < \varepsilon \tag{1}$$

This means that $x^n - a \in \ell^\infty$. But then also $a = x^n - (x^n - a) \in \ell^\infty$.
3) From (1) we obtain that for all $n \geq N$:

$$\|x^n - a\| = \sup_{k\geq 1} |x_k^n - a_k| \leq \varepsilon/2 < \varepsilon.$$

So indeed $\lim_n x^n = a$ in $\| \ \|_\infty$.
b) i) $\|x\|_* \geq 0$ because all summands $2^{-n}|x_{n+1} - x_n| \geq 0$. On the other hand,

$$\sum_n 2^{-n}|x_{n+1} - x_n| \leq \sum_n 2^{-n} 2 \sup_k |x_k| = \sum_n 2^{-n} \|x\|_\infty < \infty.$$

Here we use that $\sum_n 2^{-n} < \infty$, which explains the role of the coefficients 2^n. Note however, that if $x_n \equiv Const$, then $\|x\|_* = 0$, so $\| \ \|_*$ is at best a semi-norm.
ii) $\|\lambda x\| = \sum 2^{-n}|\lambda x_{n+1} - \lambda x_n| = \sum 2^{-n}|\lambda||x_{n+1} - x_n| = |\lambda|\|x\|_*$.
iii) $\|x+y\| = \sum 2^{-n}|x_{n+1} + y_{n+1} - (x_n + y_n)| = \sum 2^{-n}|x_{n+1} - x_n + (y_{n+1} - y_n)| \leq \sum 2^{-n}|x_{n+1} - x_n| + \sum 2^{-n}|y_{n+1} - y_n| = \|x\|_* + \|y\|_*$.
c) Take $\| \ \|_\flat = \| \ \|_* + \| \ \|_\infty$, then clearly $\| \ \|_* \leq \| \ \|_\flat$, and this time $\| \ \|_\flat$ is a proper norm, because if $x_n \equiv Const$, then $\|x\|_\flat = 0 + Const = 0$ if and only if $Const = 0$.
d) Take $d(x, y) = \|x - y\|_\infty + |x_1|$. The additional term ruins the symmetry, but not any of the other properties of a metric.