Lecture 2: § Metric and normed spaces

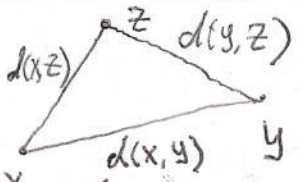
Def: Let X be a set and $d: X \times X \to \mathbb{R}$ be a function satisfying

1) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$.

2) $d(x,y) = d(y,x)$ - symmetric

3) Triangle inequality

$$d(x,y) \leq d(x,z) + d(z,y) \quad \text{for all } x,y,z \in X.$$



Then d is called a metric on X and the pair (X,d) is called a metric space. Often X is called a metric space if it is clear what metric d is chosen.

Examples:

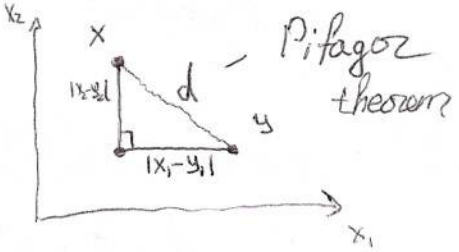
1) Let X be an arbitrary set and let

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

Then (X,d) - is a metric space

2) Usual Euclidian space $X = \mathbb{R}^n$ $(x = (x_1, \dots, x_n))$ with the standard metric

$$d_2(x,y) = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$$



- Pifagor theorem

3) $X = \mathbb{R}^n$ with the sup-metric

$$d_\infty(x,y) = \max_{i=1,n} \{|x_i - y_i|\}$$

Let us check that (X, d_∞) is a metric space:
we only need to verify the triangle inequality
We use two elementary facts

a) "triangle" inequality for real numbers

$$|a+b| \le |a|+|b|, \quad a, b \in \mathbb{R}$$

b) subadditivity of sup/max

$$\max\{A_i + B_i\} \le \max A_i + \max B_i$$

Please check!

Let $x, y, z \in X = \mathbb{R}^n$

$$d_\infty(x,y) = \max_i |x_i - y_i| = \max_i |(x_i - z_i) + (z_i - y_i)| \overset{a}{\le}$$

$$\le \max_i \{|x_i - z_i| + |z_i - y_i|\} \overset{b}{\le} \max_i |x_i - z_i| + \max_i |z_i - y_i|$$

$$= d_\infty(x,z) + d_\infty(z,y).$$

4) $\mathbb{R}^n$ with the sum-metric:

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

Please, check that $(\mathbb{R}^n, d_1)$ is a metric space.

More general metric

$$d_p(x,y) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} \qquad 1 \le p \le \infty.$$

Triangle inequality holds = Minkovski inequality.

5) The space $C[a,b]$ of continuous functions $f: [a,b] \to \mathbb{R}$ with the sup-metric

$$d(f,g) := \max_{x \in [a,b]} |f(x) - g(x)|$$

(the proof of the triangle inequality is analogous to 2)

6) Space $C[a,b]$ with the integral metric

$$d_{L^1}(f,g) := \int_a^b |f(x) - g(x)|\, dx$$

(Triangle inequality - analogous to 4)

More general L^p-metric on $C[a,b]$

$$d_{L^p}(f,g) := \left(\frac{1}{b-a} \int_a^b |f(x)-g(x)|^p \right)^{1/p} \quad 1 \le p \le \infty$$

7) If (X,d) is a metric space and $Y \subset X$ is a subset of X then (Y,d) is a metric space.

In particular, the set $\mathbb{Q}$ of rational numbers or the set $\mathbb{Z}$ of integer numbers are the metric spaces.

In Examples 2)-6) we see the important additional property - the linear structure.

Def (reminder) A set X is a linear (vector) space over $\mathbb{R}$ (or $\mathbb{C}$) if two operations $+$ and a multiplication on a real/complex number are defined

$$(x,y) \to x+y \quad , x,y \in X$$
$$(\lambda, x) \to \lambda \cdot x \quad , x \in X, \lambda \in \mathbb{R} \text{ (or } \lambda \in \mathbb{C})$$

and these two operations satisfy the following axioms

1) $x+y = y+x$

2) There exists $0 \in X$ such that $x+0 = x$

3) $(x+y)+z = x+(y+z)$

4) $(\alpha\beta)\cdot x = \alpha \cdot (\beta x)$, $1 \cdot x = x$, $0 \cdot x = 0$.

5) $\alpha(x+y) = \alpha \cdot x + \beta \cdot y$, $(\alpha+\beta)x = \alpha \cdot x + \beta \cdot x$

Examples 2)-6) are natural examples of linear spaces.

A metric d on a linear space X preserves the linear structure if

a) $d(x+z, y+z) = d(x,y)$
(homogenity of the space, invariance under the translations)

b) $d(\lambda x, \lambda y) = |\lambda| \, d(x,y)$
("enlargement" property)

If a) b) are satisfied, then

$$d(x,y) = d(x-y, 0)$$

Let $\|x\| := d(x,0)$ is called norm.

Then the properties of the metric d read

1) $\|X\| \geq 0$ and $\|X\| = 0$ if and only if $X = 0$

2) Triangle inequality $\|X+y\| \leq \|X\| + \|y\|$

3) $\|\lambda X\| = |\lambda| \cdot \|X\|$

Def.: A function $\|\cdot\|$ on a linear space X is a norm if properties 1)–3) are satisfied

Then $(X, \|\cdot\|)$ is called a normed space

Examples 2)–6) are the examples of normed spaces please, check!

Other natural examples (spaces of sequences)

8) Space ℓ^∞ of bounded sequences $X = (x_1, x_2, \dots)$ with $\sup_{i \in \mathbb{N}} |x_i| < \infty$.

The linear structure

$$X + y := (x_1 + y_1, \dots)$$
$$\lambda X = (\lambda x_1, \lambda x_2, \dots)$$

and the norm

$$\|X\|_{\ell^\infty} := \sup_{i \in \mathbb{N}} |x_i|$$

Compare with Ex. 3) and 5)

9) Space ℓ^1 of summable sequences $X = (x_1, \dots)$

$$\sum_i |x_i| < \infty$$

with the norm

$$\|X\|_{\ell^1} := \sum_{i=1}^{\infty} |x_i|$$

compare with Ex. 4) and 6).

More general: the space ℓ^p is determined by the norm

$$\|X\|_{\ell^p} = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}, \quad 1 \leq p \leq \infty.$$

Lecture 3: Limits and continuity in metric spaces

Case $X=\mathbb{R}$ (known)

a sequence $a_n \in \mathbb{R}$ of real numbers converges to $a_0 \in \mathbb{R}$ if

for any $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that

(*) $|a_n - a_0| < \varepsilon$

for all $n > N(\varepsilon)$

$d(x,y)=|x-y|$ in $\mathbb{R}$!

Case (X,d) is an abstract metric space (new).

a sequence $a_n \in X$ converges to $a_0 \in X$ if

for any $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that

$d(a_n, a_0) < \varepsilon$ (**)

for all $n > N(\varepsilon)$

The distance $d(x,y)=|x-y|$ in $\mathbb{R}$ is naturally replaced by the distance d ~~deter~~ used for defining the metric space X.

One more natural definition:

Def. Let (X,d) be a metric space. A **Ball** of radius R centered at $x_0 \in X$ is defined by

$$B_R(x_0) = \{ x \in X,\ d(x,x_0) < \varepsilon \}$$

Example I In the particular case $X=\mathbb{R}^3$, $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$

$B_R(x_0)$ is a usual R-Ball centered at x_0.

Using that notation conditions (*) and (**) can be written in the form

$$a_n \in B_\varepsilon(a_0) \text{ for all } n > N(\varepsilon).$$

In other words, $\boxed{\lim_{n\to\infty} a_n = a_0}$

if for any $\varepsilon>0$ all a_n except of at most finitely many are inside of the ε-Ball $B_\varepsilon(x_0)$

Example: ~~Let (X,d)~~ Let X be an arbitrary set and $d(x,y)=\begin{cases}1, & x\neq y\\ 0, & x=y\end{cases}$ then $a_n \xrightarrow[n\to\infty]{} a_0$ if and only if $a_n \equiv a_0$ for all $n>N$. Indeed, $B_\varepsilon(a_0)=\{a_0\}$ if $\varepsilon<1$.

Continuity

(II)

Case $X=\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ (usual functions, known)

f is continuous at $x_0\in\mathbb{R}$ if $\forall\,\varepsilon>0\ \exists\,\delta=\delta(\varepsilon)>0$ such that

$$|f(x)-f(x_0)|<\varepsilon$$

if

$$|x-x_0|<\delta$$

Case (X,d) and (Y,d) are two abstract metric spaces and $f:X\to Y$ (new).

$f:X\to Y$ is continuous at $x_0\in X$ if $\forall\,\varepsilon>0\ \exists\,\delta=\delta(\varepsilon)$ such that

$$d(f(x),f(x_0))<\varepsilon \quad (\text{distance in } Y) \qquad \Leftrightarrow \quad f(x)\in B_\varepsilon(f(x_0))$$

if

$$d(x,x_0)<\delta \quad (\text{distance in } X) \qquad \Leftrightarrow \quad x\in B_\delta(x_0)$$

Again: the only change is replacing of the "concrete" distance by the "abstract" one.

Theorem: A fuction $f:X\to Y$ is continuous at $x_0\in X$ *if and only if* for any sequence $x_n\in X$ such that $x_n\to x_0$, we have $f(x_n)\to f(x_0)$ (in Y).

In other words, we can change the order of f and $\lim$:

$$\lim_{n\to\infty} f(x_n)=f(x_0)=f\left(\lim_{n\to\infty} x_n\right) \qquad (***)$$

Proof: ($\Rightarrow$) Let f be continuous at x_0. Let us prove $(***)$. Let x_n be a sequence in X such that $x_n\to x_0$. We need to prove that $f(x_n)\to f(x_0)$. Take arbitrary $\varepsilon>0$. Then, by continuity $\exists\,\delta=\delta(\varepsilon)$ such that $f(x_n)\in B_\varepsilon(f(x_0))$ if $x_n\in B_\delta(x_0)$. since $x_n\to x_0$ then $\exists\,N=N(\delta(\varepsilon))=N(\varepsilon)$ such that $x_n\in B_\delta(x_0)$ if $n>N(\varepsilon)\Rightarrow (***)$.

($\Leftarrow$) Let f be such that $(***)$ holds for every convergent sequence. need to prove continuity. By contradiction: assume that f is NOT continuous. Then $\exists\,\varepsilon_0>0$ such that, for any $\delta>0\ \exists\,x(\delta)\in B_\delta(x_0)$ such that $f(x(\delta))\notin B_{\varepsilon_0}(f(x_0))$. Take $x_n=x(\frac{1}{n})$. Then $x_n\to x_0$ in X (since $d(x_n,x_0)<\frac{1}{n}$), but $f(x_n)\not\to f(x_0)$ (since $d(f(x_n),f(x_0))\geq\varepsilon_0$). Contradiction. $\square$

Equivalent formulation of continuity: f is continuous in x_0 if $\forall\,\varepsilon>0\ \exists\,\delta=\delta(\varepsilon)>0$ such that

$$f(B_\delta(x_0))\subset B_\varepsilon(f(x_0))$$

Ex: What functions $f:X\to X$ are continuous if (X,d) is a metric space of Example I?

Lecture 4: Convergence and continuity in finite-dimensional vector spaces.

Def: Let V be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there are positive constants C_1 and C_2 such that

$$C_2\|x\|_1 \le \|x\|_2 \le C_1\|x\|_1 \quad \text{for all } x \in V.$$

($\|\cdot\|_1 \sim \|\cdot\|_2$) - notation

Lemma Let V be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms. Then a sequence $x_n \to x_0$ in $(V, \|\cdot\|_1)$ iff it converges to x_0 in $(V, \|\cdot\|_2)$.

Proof: Let $x_n \to x_0$ in $(V, \|\cdot\|_1)$. Then $\forall \varepsilon > 0$ all x_n except of finitely many satisfy $\|x_n - x_0\|_1 < \varepsilon$

Since the norms are equivalent

$$\Rightarrow \quad \|x_n - x_0\|_2 < C_1\varepsilon \Rightarrow x_n \to x_0 \text{ in } (V, \|\cdot\|_2). \quad \square$$

Corollary: The same is true for continuous functions!

Theorem (main) Let V be a finite-dimensional vector space (over $\mathbb{R}$ or $\mathbb{C}$). Then any two norms on V are equivalent.

Proof: **Part 1**: V is equivalent to $\mathbb{R}^n$ as a vector space (known). Indeed, let $n = \dim V$ and let $\{e_1, \dots e_n\}$ be a basis in V. Then any element $v \in V$ has the form

(*) $$v = x_1 e_1 + \dots + x_n e_n$$

and representation (*) is unique. Thus, we have one-to-one correspondence

$$V \ni v = x_1 e_1 + \dots + x_n e_n \leftrightarrow (x_1, \dots, x_n) \in \mathbb{R}^n.$$

+ the linear structure is preserved.

Thus, without loss of generality, we may consider $V = \mathbb{R}^n$ only. Then, any norm $\|v\| = \|x_1 e_1 + \dots + x_n e_n\| = f(x_1, \dots, x_n)$ is a real valued function on $\mathbb{R}^n$.

Part II: any norm on $\mathbb{R}^n$ is a continuous function on $\mathbb{R}^n$ with the standard norm $\|x\|^2 = x_1^2 + \dots + x_n^2$.

Indeed let $\|x\|_*$ be arbitrary norm on $\mathbb{R}^n$. By triangle inequality, we have

$$\|x\|_* \le \|x-y\|_* + \|y\|_* \quad \Rightarrow \quad \big|\|x\|_* - \|y\|_*\big| \le \|x-y\|_*$$
$$\|y\|_* \le \|x-y\|_* + \|x\|_*$$

Thus,

$$\big|\|x\|_* - \|y\|_*\big| \le \|x-y\|_* = \|(x_1-y_1)e_1 + \dots + (x_n-y_n)e_n\|_* \le$$

$$\le |x_1-y_1|\cdot\|e_1\|_* + \dots + |x_n-y_n|\cdot\|e_n\|_* \le C\left(|x_1-y_1| + \dots + |x_n-y_n|\right) \le$$

$$\le Cn\sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2} = Cn\,\|x-y\|.$$

In a fact, we have even proved that

$$\boxed{\|x\|_* \le Cn\,\|x\|}$$ ← the standard norm on $\mathbb{R}^n$.

Part III: opposite inequality. Let us consider a unit sphere S_1 in $\mathbb{R}^n$ (with a standard norm) and let

$$\boxed{c := \inf_{x \in S_1} \|x\|_*}$$

Then

$$\|x\|_* \ge c\,\|x\| \quad \text{for all } x \in S_1 \text{ and}$$

(since $\|\lambda x\| = |\lambda|\,\|x\|$ and the same is true for $\|\cdot\|_*$) the inequality holds for all $x \in \mathbb{R}^n$.

Thus, we only need to check that $\underline{c > 0}$.

By the inf definition $\exists$ a sequence $x_n \in S_1$ such that $\|x_n\|_* \to c$ as $n \to \infty$

$\{x_n\}$ is a bounded sequence in $\mathbb{R}^n$ ⇒ By Heine-Borel lemma $\exists$ a convergent subsequence $x_{n_k} \to x_0$, $x_0 \in S_1$ (By continuity of $\|\cdot\|$). In particular, $x_0 \neq 0$.

Then, by continuity of $\|\cdot\|_*$

$$c = \lim_{k\to\infty} \|x_{n_k}\|_* = \Big\|\lim_{k\to\infty} x_{n_k}\Big\|_* = \|x_0\|_* > 0$$

□

Corollary: the convergence in $\mathbb{R}^n$ (finite-dimensional vector space) is "uniquelly determined".

$\Rightarrow$ our theory is "useless" is useless in the finite-dimensional case!

But it is NOT about the finite-dimensional case, but about the spaces of functions which are infinite-dimensional!!!

Example: $X = C[0,1]$ - space of continuous functions on $[0,1]$, $f: [0,1] \to \mathbb{R}$.

Consider two norms on X

first - the sup-norm

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|$$

second - the L^1- integral norm

$$\|f\|_{L^1} := \int_0^1 |f(x)|\,dx$$

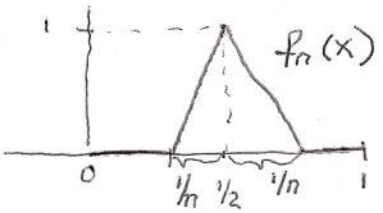
Then, obviously

$$\|f\|_{L^1} \le \|f\|_\infty$$

But these norms ARE NOT equivalent!!!

indeed

let $f_n(x)$ be a sequence of functions constructed as follows:



Then

$$\|f_n\|_\infty = 1 \quad \text{but} \quad \|f_n\|_{L^1} = \frac{1}{n} \to 0.$$

Main reason: The Heine-Borel lemma does not true for the inf-dim case (bounded sequence does not necessarily contains a convergent subsequence). - major difficulty of the inf-dim case.

As a result: different types of convergences, different continuous functions, etc.

Proper choice of the norm/metric becomes crucial, different problems require different choices

The "universal choice" does not exist!