Lecture 5: Open and closed sets in metric spaces.

Def: Let (X,d) be a metric space. A set $V\subset X$ is open if for any $x\in V$ $\exists\, \varepsilon>0$ such that

$$\boxed{B_\varepsilon(x)\subset V.}$$

2) A point $x_0\in X$ is a limit point of V if $\exists$ a sequence $x_n\in V$ such that

$$\boxed{x_n\to x_0 \quad \text{as } n\to\infty}$$

A closure $\overline{V}$ of the set $V\subset X$ is a set of all limit points of V. A set V is closed if V contains all its limit points:

$$\boxed{V=\overline{V}}$$

A ~~set int V is~~ point $x_0\in V$ is an interior point of V if $\exists\, \varepsilon>0$ such that

$$\boxed{B_\varepsilon(x_0)\subset V}$$

A set of all interior points of V is denoted by $\operatorname{int} V$.

A set $\partial V:=\overline{V}\setminus \operatorname{int} V$ is a boundary of the set V.

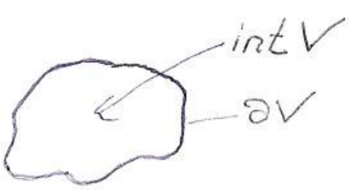
Proposition: 1) for any set V, $\overline{V}$ - is closed and $\operatorname{int} V$ is open

2) the boundary ∂V consists of all points $x_0\in X$ such that $\exists\, \varepsilon=\varepsilon(x_0)>0$ and

$$\boxed{B_\varepsilon(x_0)\cap V\neq\emptyset} \quad \boxed{B_\varepsilon(x_0)\cap(X\setminus V)\neq\emptyset}$$

please, check!

Example:



Proposition: Let (X,d) be a metric space. Then

1) A ball $B_\varepsilon(x_0)$ is an open set, for any $\varepsilon>0$ and $x_0\in X$

2) A closed ball $\boxed{\overline{B}_\varepsilon(x_0):=\{x\in X,\ d(x_0,x)\le\varepsilon\}}$ is a closed set

3) A sphere $S_\varepsilon(x_0):=\{x\in X,\ d(x_0,x)=\varepsilon\}$ is a boundary of $B_\varepsilon(x_0)$

$$\boxed{S_\varepsilon(x_0)=\partial B_\varepsilon(x_0)=\partial \overline{B}_\varepsilon(x_0)}$$

Proof: 1) Let $y \in B_\varepsilon(x_0) \Rightarrow d(x_0, y) = \varepsilon_0 < \varepsilon$

Consider $B_{\varepsilon-\varepsilon_0}(y)$. By the triangle inequality

$$d(z, x_0) \le d(y, x_0) + d(z, y) < \varepsilon_0 + \varepsilon - \varepsilon_0 = \varepsilon, \quad \forall z \in B_{\varepsilon-\varepsilon_0}(y)$$

$\Rightarrow$ $B_{\varepsilon-\varepsilon_0}(y) \subset B_\varepsilon(x_0) \Rightarrow B_\varepsilon(x_0)$ is open.

2) Let y_0 be a limit point of $\overline{B}_\varepsilon(x_0)$. Then $\exists \{x_n\}$ $\boxed{d(x_n, x_0) \le \varepsilon}$ and $\boxed{x_n \to y_0}$. We need to prove that $\boxed{d(y_0, x_0) \le \varepsilon}$. Sufficient to check that the function $x \mapsto d(x, x_0)$ is continuous.

By the triangle inequality

$$\left.\begin{array}{l} d(x_0, x) \le d(x_0, y) + d(x, y) \\ d(x_0, y) \le d(x_0, x) + d(x, y) \end{array}\right| \Rightarrow \boxed{|d(x_0, x) - d(x_0, y)| \le d(x, y)}$$

$\Rightarrow$ d is continuous!

3) is evident

Ex: Please, check that

$$\boxed{|d(x, y) - d(v, w)| \le d(x, v) + d(y, w)}$$

and, therefore, $d(x, y)$ is continuous with respect to both arguments.

Properties of open sets:

topology on X = class of open sets satisfying 1)2)3).

1) X and $\emptyset$ are open sets

2) a union of any number of open sets is open

X_α - are open for $\alpha \in A$, then

$$\boxed{\bigcup_{\alpha \in A} X_\alpha \text{ - is open}}$$

3) Intersection of any finite number of open sets is open

X_i, $i = 1, \dots, n$ are open

$$\Rightarrow \boxed{\bigcap_{i=1}^{n} X_i \text{ - is open}}$$

Example: $X = \mathbb{R}$ $\quad X_n = (-\frac{1}{n}, \frac{1}{n})$ are open sets, but

$$\bigcap_{n=1}^{\infty} X_n = \{0\} \text{ is not open!}$$

"finite number" is essential in 3).

Lemma Let (X,d) be a metric space. Then a set $V \subset X$ is open iff the complement $X \setminus V$ is closed

Proof: Let V be open. We need to check that $X\setminus V$ is closed. Let $x_n \in X\setminus V$ be a sequence converging to some $x_0 \in X$. We need to prove that $x_0 \in X\setminus V$.
By contradiction, assume that $x_0 \notin X\setminus V$

$\Rightarrow \boxed{x_0 \in V}$, but V is open $\Rightarrow \exists \varepsilon > 0,\ B_\varepsilon(x_0) \in V$

$x_n \to x_0 \Rightarrow \boxed{x_n \in B_\varepsilon(x_0),\ n \geq N} \Rightarrow x_n \in V,\ n \geq N \Rightarrow$

$\boxed{x_n \notin X\setminus V}$ - contradiction.

Let now $X\setminus V$ be closed. Need to prove V is open. Again, by contradiction. Assume that V is not open $\Rightarrow \exists x_0 \in V$ such that $\forall \varepsilon > 0$ $\boxed{\exists x_\varepsilon \notin V}$ but $\boxed{x_\varepsilon \in B_\varepsilon(x_0)}$
Take $\varepsilon = \frac{1}{n}$. Then $\boxed{x_{1/n} \in X\setminus V}$ and $\boxed{x_{1/n} \to x_0 \in V\ (\notin X\setminus V)}$
$\Rightarrow X\setminus V$ is not closed. Contradiction. $\square$

Properties of closed sets:

I) X and $\emptyset$ are closed

II) An intersection of any number of closed sets is closed:

$$\boxed{X_\alpha \subset X \text{ are closed}, \alpha \in A} \Rightarrow \boxed{\bigcap_{\alpha \in A} X_\alpha \text{ is closed}}$$

III A union of a FINITE number of closed sets is closed

$$\boxed{X_i \subset X,\ i = 1, \dots, n \text{ are closed}} \Rightarrow \boxed{\bigcup_{i=1}^{n} X_i \text{ is closed}}$$

Please check!
Hint: You may reduce that to the case of open sets using the lemma and De Morgan's laws:

$$\boxed{X \setminus \left(\bigcup_\alpha X_\alpha\right) = \bigcap_\alpha (X \setminus X_\alpha)}$$

$$\boxed{X \setminus \left(\bigcap_\alpha X_\alpha\right) = \bigcup_\alpha (X \setminus X_\alpha)}$$

Convergence and continuity via open sets. (IV)

Aim: to reformulate the definitions in terms of open sets instead of metrics.

Def 1 (metric) $\boxed{x_n \to x_o}$ iff $\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ such that

$$\boxed{x_n \subset \underbrace{B_\varepsilon(x_o)}_{\text{open set}}, \ n \geq N}$$

Def 2 (topologic) $\boxed{x_n \to x_o}$ iff $\forall$ open set $V \subset X$, $x_o \in V$ $\exists N = N(V)$ such that

$$\boxed{x_n \subset V, \ n \geq N}$$

Def.: Let $x_o \in X$, V is an (open) neighborhood of x_o if $\boxed{x_o \in V}$ and $\boxed{V \text{ is open}}$

Lemma: In any METRIC space (X, d)

$$\text{Def. 1} \Leftrightarrow \text{Def. 2}$$

Proof: Since $B_\varepsilon(x_o)$ is open, Def. 2 $\Rightarrow$ Def. 1

Let us prove that Def. 1 $\Rightarrow$ Def. 2

Let $x_n \to x_o$ in the sense of Def. 1 and let V be an arbitrary neighborhood of x_o. $\boxed{V \text{ is open}} + \boxed{x_o \in V}$
$\Rightarrow \exists \varepsilon > 0$ such that $\boxed{B_\varepsilon(x_o) \subset V}$ from Def. 1
$\Rightarrow \exists N = N(\varepsilon)$ such that

$$\boxed{x_n \in B_\varepsilon(x_o) \subset V} \quad n \geq N$$

$\Rightarrow$ Def. 2 holds $\square$

Continuity:

Def. 1 (metric) $f: X \to Y$ is continuous at $x_o \in X$ iff $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0$ such that

$$\boxed{f(B_\delta(x_o)) \subset B_\varepsilon(f(x_o))}$$

Def. 2 (topologic): $f: X \to Y$ is continuous at $x_o \in X$ iff $\forall$ neighborhood $V \subset Y$ of $f(x_o)$ $\exists$ neighborhood $W \subset X$ of x_o such that

$$\boxed{f(W) \subset V}$$

Lemma: In any Metric space

$$\text{Def. 1} \Leftrightarrow \text{Def. 2}$$

Proof:

Def. 1 $\Rightarrow$ Def. 2

Let f be continuous at $x_0 \in X$ in the sense of Def. 1 and let $V \subset Y$ be a neighborhood of $f(x_0) \Rightarrow \exists \varepsilon > 0$ such that

$$\boxed{B_\varepsilon(f(x_0)) \subset V}$$

Def. 1 $\Rightarrow \exists \delta > 0$ $\boxed{f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \subset V}$

$\Rightarrow$ Def. 2 holds with $\boxed{W = B_\delta(x_0)}$

Def. 2 $\Rightarrow$ Def. 1 Let f be continuous in the sense of Def. 2. Take $\boxed{V = B_\varepsilon(f(x_0))}$ - neighborhood of $f(x_0)$ $\Rightarrow \exists\, W \subset X$ - neighborhood of x_0 such that

$$\boxed{f(W) \subset B_\varepsilon(f(x_0))}$$

W is open $\Rightarrow \exists \delta > 0$

$$\boxed{B_\delta(x_0) \subset W}$$

$\Rightarrow f(B_\delta(x_0)) \subset f(W) \subset B_\varepsilon(f(x_0)) \Rightarrow$ Def. 1 holds $\square$

$f: X \to Y$ is continuous on X $(f \in C(X,Y))$ = f is continuous at every $x_0 \in X$.

Theorem: $f \in C(X,Y) \Leftrightarrow$ the inverse image of any open set is open:

$$\boxed{V \subset Y \text{ is open}} \Rightarrow \boxed{f^{-1}(V) \subset X \text{ is open}}$$

Proof: $\Rightarrow$ let f be continuous and V be open. Need to check that $f^{-1}(V)$ is open. Let $\boxed{x_0 \in f^{-1}(V)} \Rightarrow \boxed{f(x_0) \in V}$ + V is open $\Rightarrow \exists\, \varepsilon > 0$ such that $\boxed{B_\varepsilon(f(x_0)) \subset V}$ + continuity $\Rightarrow$

$\Rightarrow \exists\, \delta > 0$ $\boxed{f(B_\delta(x_0)) \subset B_\varepsilon(f(x_0)) \subset V} \Rightarrow \boxed{B_\delta(x_0) \subset f^{-1}(V)}$

$\Leftarrow$ Let $f^{-1}(V)$ be open for any open V. Need to prove that f is continuous at every $x_0 \in X$. Use Def. 2 of continuity. For any neighborhood $V \subset Y$ of $f(x_0)$ $\boxed{W = f^{-1}(V)}$ is open and $x_0 \in W \Rightarrow W \subset X$ is a neighborhood of x_0.

But $\boxed{f(W) = f(f^{-1}(V)) \subset V} \Rightarrow f$ is continuous at $x_0 \in X$.

Why we cannot write "=" here?